2-Factor Random Effects ANOVA

Engineering Statistics II

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PART I:

2-Factor Randomized Complete Block ANOVA
(2F rcbANOVA)

Random Effects Model Assumptions

Random Effects Linear Model

F-Test Procedure

Expected Mean Squares

Point Estimators of σ^2, σ_A^2 & $\sigma_{[B]}^2$

2F rcbANOVA Random Effects Model Assumptions

Random effects means the levels of a factor are randomly selected.

Proposition

- *(**1 Desired Factor**) The sole factor of interest has I levels.*
- *(**1 Nuisance Factor**) The sole nuisance factor has J levels.*
- *(**Factor Levels are Randomly Selected**) AKA Random Effects.*
- *(**1 Measurement per Group**) Each of the IJ groups has one exp unit.*

- *(**Random Assignment within Blocks**) such that (s.t.)*
- *(**Nuisance Same in Block**) Within block, nearly same nuisance values.*
- *(**Nuisance Differs across Blocks**) Blocks differ by nuisance value.*

- *(**Independence**) All measurements on units are independent.*
- *(**Normality**) All IJ groups are approximately normally distributed.*
- *(**Equal Variances**) All IJ groups have approximately same variance.*
- *(**Factor and Block are not Interactive**)*

1DF 1NF FLaRS 1MpG | RAwB s.t. NSiB NDaB | I.N.EV FaBanI

2-Factor rcbANOVA Linear Model (Random Effects)

Random effects means the levels of a factor are randomly selected.

2F rcbANOVA Random Effects Linear Model

(I, J)	$\equiv$	(# levels of factor A, # levels of blocked nuisance factor B)
X_{ij}	$\equiv$	rv for observation at (i, j)-level of (factor A, block B)
μ	$\equiv$	Mean avg response over all levels of (factor A, block B)
$(A_i, [B]_j)$	$\equiv$	rv for effect of (i^{th}-level factor A, j^{th}-level block B)
E_{ij}	$\equiv$	Deviation from μ due to random error

<u>ASSUMPTIONS:</u>

$E_{ij} \overset{iid}{\sim}$ Normal $\left(0, \sigma^2\right)$, $\quad A_i \overset{iid}{\sim}$ Normal $\left(0, \sigma_A^2\right)$, $\quad [B]_j \overset{iid}{\sim}$ Normal $\left(0, \sigma_{[B]}^2\right)$

$A_i,\ [B]_j\ \&\ E_{ij}$ are all mutually independent of each other

$$X_{ij} = \mu + A_i + [B]_j + E_{ij}$$

$$H_0^A : \ \sigma_A^2 = 0$$
$$H_A^A : \ \sigma_A^2 > 0$$

2F rcbANOVA Random Effects F-Test

1. $\nu_A = I - 1,\ \nu_{[B]} = J - 1,\ \nu_{res} = (I-1)(J-1)$

2. $\bar{x}_{i\bullet} := \frac{1}{J}\sum_j x_{ij}, \quad \bar{x}_{\bullet j} := \frac{1}{I}\sum_i x_{ij}$
3. $\bar{x}_{\bullet\bullet} := \frac{1}{IJ}\sum_i \sum_j x_{ij}$

4. $$\begin{cases} \mathrm{SS}_{res} & := & \sum_{ij}(x_{ijk}^{res})^2 & = & \sum_i \sum_j (x_{ij} - \bar{x}_{i\bullet} - \bar{x}_{\bullet j} + \bar{x}_{\bullet\bullet})^2 \\ \mathrm{SS}_A & := & \sum_{ij}(\hat{\alpha}_i^A)^2 & = & \sum_i \sum_j (\bar{x}_{i\bullet} - \bar{x}_{\bullet\bullet})^2 \\ \mathrm{SS}_{[B]} & := & \sum_{ij}(\hat{\alpha}_j^{[B]})^2 & = & \sum_i \sum_j (\bar{x}_{\bullet j} - \bar{x}_{\bullet\bullet})^2 \end{cases}$$

(Optional) $\mathrm{SS}_{total} := \sum_{ij}(x_{ij} - \hat{\mu})^2 = \sum_i \sum_j (x_{ij} - \bar{x}_{\bullet\bullet})^2$

2F rcbANOVA Random Effects F-Test

5. $\text{MS}_A = \dfrac{\text{SS}_A}{\nu_A}, \quad \text{MS}_{[B]} = \dfrac{\text{SS}_{[B]}}{\nu_{[B]}}, \quad \text{MS}_{res} = \dfrac{\text{SS}_{res}}{\nu_{res}}$

6. $f_A = \dfrac{\text{MS}_A}{\text{MS}_{res}}, \quad f_{[B]} = \dfrac{\text{MS}_{[B]}}{\text{MS}_{res}}$

7. (if using software): $\begin{cases} p_A & := & \mathbb{P}(F > f_A) & \approx & 1 - \Phi_F(f_A; \nu_A, \nu_{res}) \\ p_{[B]} & := & \mathbb{P}(F > f_{[B]}) & \approx & 1 - \Phi_F(f_{[B]}; \nu_{[B]}, \nu_{res}) \end{cases}$

8. $\begin{cases} \text{If} \quad p_A \leq \alpha \quad \text{or} \quad f_A > f^*_{\nu_A, \nu_{res}; \alpha} & \text{then reject } H_0^A \text{, else accept } H_0^A. \\ \text{If} \quad p_{[B]} \leq \alpha \quad \text{or} \quad f_{[B]} > f^*_{\nu_{[B]}, \nu_{res}; \alpha} & \text{then the blocking was effective}^\dagger. \end{cases}$

†"Effective" means a reduced MS_{res} compared to 1F bcrANOVA.

2F rcbANOVA Random Effects (Expected Mean Squares)

Proposition

Given 2F experiment satisfying 2F rcbANOVA random effects assumptions. Then:

$$
\begin{array}{rrcl}
(i) & \mathbb{E}[MS_{res}] & = & \sigma^2 \\
(ii) & \mathbb{E}[MS_A] & = & \sigma^2 + J\sigma_A^2 \\
(iii) & \mathbb{E}[MS_{[B]}] & = & \sigma^2 + I\sigma_{[B]}^2
\end{array}
$$

2F rcbANOVA Random Effects (Point Estimators of σ^2, σ_A^2 & $\sigma_{[B]}^2$)

Proposition

Given 2F experiment satisfying 2F rcbANOVA random effects assumptions. Then:

$$
\begin{aligned}
(i) \quad & \hat{\sigma}^2 & = \quad & MS_{res} \\
(ii) \quad & \hat{\sigma}_A^2 & = \quad & (MS_A - MS_{res})/J \\
(iii) \quad & \hat{\sigma}_{[B]}^2 & = \quad & (MS_{[B]} - MS_{res})/I
\end{aligned}
$$

PART II:

2-Factor Balanced Completely Randomized ANOVA
(2F bcrANOVA)

Random Effects Model Assumptions

Random Effects Linear Model

F-Test Procedure

Expected Mean Squares

Point Estimators of σ^2, σ_A^2, σ_B^2 & σ_{AB}^2

2F bcrANOVA Random Effects Model Assumptions

Random effects means the levels of a factor are randomly selected.

Proposition

(2F bcrANOVA Random Effects Model Assumptions)

- *(**2 Desired Factors**) Factor A has I levels & Factor B has J levels.*
- *(**Factor Levels are Randomly Selected**) AKA Random Effects.*
- *(**Factors are Crossed**) IJ groups – one per (i, j)-level factor combination.*
- *(**Balanced Replication in Groups**) Each group has $K > 1$ units.*
- *(**Distinct Exp. Units**) All IJK units are distinct from each other.*

- *(**Random Assignment across Groups**)*

- *(**Independence**) All measurements on units are independent.*
- *(**Normality**) All groups are approximately normally distributed.*
- *(**Equal Variances**) All groups have approximately same variance.*

Mnemonic: ***2DF FLaRS FaC BRiG DEU | RAaG | I.N.EV***

Random effects means the levels of a factor are randomly selected.

2F bcrANOVA Random Effects Linear Model

(I,J)	$\equiv$	(# levels of factor A, # levels of factor B)
K	$\equiv$	# observations (replications) at each (i,j)-level of factors A & B
X_{ijk}	$\equiv$	rv for k^{th} observation at (i,j)-level of factors A & B
μ	$\equiv$	Mean average response over all levels of factors A & B
(A_i, B_j)	$\equiv$	(rv for effect of i^{th}-level factor A, rv for effect of j^{th}-level factor B)
AB_{ij}	$\equiv$	rv for interaction between (i,j)-level factors A & B
E_{ijk}	$\equiv$	Deviation from μ due to random error

ASSUMPTIONS:

$E_{ijk} \overset{iid}{\sim}$ Normal $\left(0, \sigma^2\right)$, $A_i \overset{iid}{\sim}$ N $\left(0, \sigma_A^2\right)$, $B_j \overset{iid}{\sim}$ N $\left(0, \sigma_B^2\right)$, $AB_{ij} \overset{iid}{\sim}$ N $\left(0, \sigma_{AB}^2\right)$

A_i, B_j, AB_{ij} & E_{ijk} are all mutually independent of each other

$$X_{ijk} = \mu + A_i + B_j + AB_{ij} + E_{ijk}$$

$H_0^A : \sigma_A^2 = 0$	$H_0^B : \sigma_B^2 = 0$	$H_0^{AB} : \sigma_{AB}^2 = 0$
$H_A^A : \sigma_A^2 > 0$	$H_A^B : \sigma_B^2 > 0$	$H_A^{AB} : \sigma_{AB}^2 > 0$

2F bcrANOVA Random Effects F-Test

1. $\nu_A = I-1,\ \nu_B = J-1,\ \nu_{AB} = (I-1)(J-1),\ \nu_{res} = IJ(K-1)$
2. $\bar{x}_{i\bullet\bullet} := \frac{1}{JK}\sum_j\sum_k x_{ijk},\quad \bar{x}_{\bullet j\bullet} := \frac{1}{IK}\sum_i\sum_k x_{ijk},\quad \bar{x}_{ij\bullet} := \frac{1}{K}\sum_k x_{ijk}$
3. $\bar{x}_{\bullet\bullet\bullet} := \frac{1}{IJK}\sum_i\sum_j\sum_k x_{ijk}$
4. $$\begin{cases} \mathrm{SS}_{res} & := & \sum_{ijk}(x_{ijk}^{res})^2 & = & \sum_i\sum_j\sum_k(x_{ijk}-\bar{x}_{ij\bullet})^2 \\ \mathrm{SS}_A & := & \sum_{ijk}(\hat{\alpha}_i^A)^2 & = & \sum_i\sum_j\sum_k(\bar{x}_{i\bullet\bullet}-\bar{x}_{\bullet\bullet\bullet})^2 \\ \mathrm{SS}_B & := & \sum_{ijk}(\hat{\alpha}_j^B)^2 & = & \sum_i\sum_j\sum_k(\bar{x}_{\bullet j\bullet}-\bar{x}_{\bullet\bullet\bullet})^2 \\ \mathrm{SS}_{AB} & := & \sum_{ijk}(\hat{\gamma}_{ij}^{AB})^2 & = & \sum_i\sum_j\sum_k(\bar{x}_{ij\bullet}-\bar{x}_{i\bullet\bullet}-\bar{x}_{\bullet j\bullet}+\bar{x}_{\bullet\bullet\bullet})^2 \end{cases}$$

(Optional) $\mathrm{SS}_{total} := \sum_{ijk}(x_{ijk}-\hat{\mu})^2 = \sum_i\sum_j\sum_k(x_{ijk}-\bar{x}_{\bullet\bullet\bullet})^2$

2F bcrANOVA Random Effects F-Test

5. $$\text{MS}_A = \frac{\text{SS}_A}{\nu_A}, \quad \text{MS}_B = \frac{\text{SS}_B}{\nu_B}, \quad \text{MS}_{AB} = \frac{\text{SS}_{AB}}{\nu_{AB}}, \quad \text{MS}_{res} = \frac{\text{SS}_{res}}{\nu_{res}}$$

6. $$f_A = \frac{\text{MS}_A}{\text{MS}_{AB}}, \quad f_B = \frac{\text{MS}_B}{\text{MS}_{AB}}, \quad f_{AB} = \frac{\text{MS}_{AB}}{\text{MS}_{res}}$$

7. (if using software): $$\begin{cases} p_A & := & \mathbb{P}(F > f_A) & \approx & 1 - \Phi_F(f_A; \nu_A, \nu_{res}) \\ p_B & := & \mathbb{P}(F > f_B) & \approx & 1 - \Phi_F(f_B; \nu_B, \nu_{res}) \\ p_{AB} & := & \mathbb{P}(F > f_{AB}) & \approx & 1 - \Phi_F(f_{AB}; \nu_{AB}, \nu_{res}) \end{cases}$$

8. $$\begin{cases} \text{If} & p_A \leq \alpha & \text{or} & f_A > f^*_{\nu_A, \nu_{res}; \alpha} & \text{then reject} & H_0^A & \text{else accept} & H_0^A \\ \text{If} & p_B \leq \alpha & \text{or} & f_B > f^*_{\nu_B, \nu_{res}; \alpha} & \text{then reject} & H_0^B & \text{else accept} & H_0^B \\ \text{If} & p_{AB} \leq \alpha & \text{or} & f_{AB} > f^*_{\nu_{AB}, \nu_{res}; \alpha} & \text{then reject} & H_0^{AB} & \text{else accept} & H_0^{AB} \end{cases}$$

2F bcrANOVA (Expected Mean Squares)

Proposition

Given 2F experiment satisfying 2F bcrANOVA random effects assumptions. Then:

$$
\begin{aligned}
(i) \quad & \mathbb{E}[MS_{res}] &=& \ \sigma^2 \\
(ii) \quad & \mathbb{E}[MS_A] &=& \ \sigma^2 + K\sigma_{AB}^2 + JK\sigma_A^2 \\
(iii) \quad & \mathbb{E}[MS_B] &=& \ \sigma^2 + K\sigma_{AB}^2 + IK\sigma_B^2 \\
(iv) \quad & \mathbb{E}[MS_{AB}] &=& \ \sigma^2 + K\sigma_{AB}^2
\end{aligned}
$$

2F bcrANOVA Random Effects (Point Estimators of σ^2, σ_A^2, σ_B^2 & σ_{AB}^2)

Proposition

Given 2F experiment satisfying 2F bcrANOVA random effects assumptions. Then:

$$
\begin{array}{lrcl}
(i) & \hat{\sigma}^2 & = & MS_{res} \\
(ii) & \hat{\sigma}_A^2 & = & (MS_A - MS_{AB})/JK \\
(iii) & \hat{\sigma}_B^2 & = & (MS_B - MS_{AB})/IK \\
(iv) & \hat{\sigma}_{AB}^2 & = & (MS_{AB} - MS_{res})/K
\end{array}
$$

Fin.