2-Factor Mixed Effects ANOVA

Engineering Statistics II

Josh Engwer

TTU

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PART I:

2-Factor Randomized Complete Block ANOVA
(2F rcbANOVA)

Mixed Effects Model Assumptions

Mixed Effects Linear Model

F-Test Procedure

Expected Mean Squares

Point Estimators of σ^2 & σ_A^2

2F rcbANOVA Mixed Effects Model Assumptions

Mixed effects means one fixed factor and one random factor.

Proposition

- (***1 Desired Factor***) *The sole factor of interest has* I *levels.*
- (***1 Nuisance Factor***) *The sole nuisance factor has* J *levels.*
- (***Factor Levels are Mixed in Nature***) *AKA Mixed Effects.*
- (***1 Measurement per Group***) *Each of the* IJ *groups has one exp unit.*

- (***Random Assignment within Blocks***) *such that (s.t.)*
- (***Nuisance Same in Block***) *Within block, nearly same nuisance values.*
- (***Nuisance Differs across Blocks***) *Blocks differ by nuisance value.*

- (***Independence***) *All measurements on units are independent.*
- (***Normality***) *All* IJ *groups are approximately normally distributed.*
- (***Equal Variances***) *All* IJ *groups have approximately same variance.*
- (***Factor and Block are not Interactive***)

1DF 1NF FLaMiN 1MpG | RAwB s.t. NSiB NDaB | I.N.EV FaBanI

2-Factor rcbANOVA Linear Model (Mixed Effects)

Mixed effects means one fixed factor and one random factor.

2F rcbANOVA Mixed Effects Linear Model

(I, J)	$\equiv$	(# levels of factor A, # levels of blocked nuisance factor B)
X_{ij}	$\equiv$	rv for observation at (i, j)-level of (factor A, block B)
μ	$\equiv$	Mean avg response over all levels of (factor A, block B)
$(A_i, \alpha_j^{[B]})$	$\equiv$	(rv for effect of i^{th}-level factor A), (effect of j^{th}-level block B)
E_{ij}	$\equiv$	Deviation from μ due to random error

ASSUMPTIONS:

$$E_{ij} \overset{iid}{\sim} \text{Normal}\left(0, \sigma^2\right), \quad A_i \overset{iid}{\sim} \text{Normal}\left(0, \sigma_A^2\right)$$

A_i & E_{ij} are all mutually independent of each other

$$X_{ij} = \mu + A_i + \alpha_j^{[B]} + E_{ij}$$

$$H_0^A : \sigma_A^2 = 0$$
$$H_A^A : \sigma_A^2 > 0$$

1. $\nu_A = I - 1,\ \nu_{[B]} = J - 1,\ \nu_{res} = (I-1)(J-1)$

2. $\bar{x}_{i\bullet} := \frac{1}{J}\sum_j x_{ij},\quad \bar{x}_{\bullet j} := \frac{1}{I}\sum_i x_{ij}$
3. $\bar{x}_{\bullet\bullet} := \frac{1}{IJ}\sum_i \sum_j x_{ij}$

4. $$\begin{cases} \mathrm{SS}_{res} & := & \sum_{ij}(x^{res}_{ijk})^2 & = & \sum_i \sum_j (x_{ij} - \bar{x}_{i\bullet} - \bar{x}_{\bullet j} + \bar{x}_{\bullet\bullet})^2 \\ \mathrm{SS}_{A} & := & \sum_{ij}(\hat{\alpha}^A_i)^2 & = & \sum_i \sum_j (\bar{x}_{i\bullet} - \bar{x}_{\bullet\bullet})^2 \\ \mathrm{SS}_{[B]} & := & \sum_{ij}(\hat{\alpha}^{[B]}_j)^2 & = & \sum_i \sum_j (\bar{x}_{\bullet j} - \bar{x}_{\bullet\bullet})^2 \end{cases}$$

(Optional) $\mathrm{SS}_{total} := \sum_{ij}(x_{ij} - \hat{\mu})^2 = \sum_i \sum_j (x_{ij} - \bar{x}_{\bullet\bullet})^2$

2F rcbANOVA Mixed Effects F-Test

5. $$\text{MS}_A = \frac{\text{SS}_A}{\nu_A}, \quad \text{MS}_{[B]} = \frac{\text{SS}_{[B]}}{\nu_{[B]}}, \quad \text{MS}_{res} = \frac{\text{SS}_{res}}{\nu_{res}}$$

6. $$f_A = \frac{\text{MS}_A}{\text{MS}_{res}}, \quad f_{[B]} = \frac{\text{MS}_{[B]}}{\text{MS}_{res}}$$

7. (if using software): $$\begin{cases} p_A := \mathbb{P}(F > f_A) \approx 1 - \Phi_F(f_A; \nu_A, \nu_{res}) \\ p_{[B]} := \mathbb{P}(F > f_{[B]}) \approx 1 - \Phi_F(f_{[B]}; \nu_{[B]}, \nu_{res}) \end{cases}$$

8. $$\begin{cases} \text{If } p_A \leq \alpha \text{ or } f_A > f^*_{\nu_A,\nu_{res};\alpha} & \text{then reject } H_0^A\text{, else accept } H_0^A. \\ \text{If } p_{[B]} \leq \alpha \text{ or } f_{[B]} > f^*_{\nu_{[B]},\nu_{res};\alpha} & \text{then the blocking was effective}^\dagger. \end{cases}$$

†"Effective" means a reduced MS_{res} compared to 1F bcrANOVA.

2F rcbANOVA Mixed Effects (Expected Mean Squares)

Proposition

Given 2F experiment satisfying 2F rcbANOVA mixed effects assumptions. Then:

$$
\begin{aligned}
(i) \quad & \mathbb{E}[MS_{res}] &=& \quad \sigma^2 \\
(ii) \quad & \mathbb{E}[MS_A] &=& \quad \sigma^2 + J\sigma_A^2 \\
(iii) \quad & \mathbb{E}[MS_{[B]}] &=& \quad \sigma^2 + \frac{I}{J-1}\sum_j (\alpha_j^{[B]})^2
\end{aligned}
$$

2F rcbANOVA Mixed Effects (Point Estimators of σ^2 & σ_A^2)

Proposition

Given 2F experiment satisfying 2F rcbANOVA mixed effects assumptions. Then:

$$(i) \quad \hat{\sigma}^2 = MS_{res}$$

$$(ii) \quad \hat{\sigma}_A^2 = (MS_A - MS_{res})/J$$

PART II:

2-Factor Balanced Completely Randomized ANOVA
(2F bcrANOVA)

Mixed Effects Model Assumptions

Mixed Effects Linear Model

F-Test Procedure

Expected Mean Squares

Point Estimators of σ^2, σ_A^2 & σ_{AB}^2

2F bcrANOVA Mixed Effects Model Assumptions

Mixed effects means one fixed factor and one random factor.

Proposition

(2F bcrANOVA Mixed Effects Model Assumptions)

- *(**2 Desired Factors**) Factor A has I levels & Factor B has J levels.*
- *(**Factor Levels are Mixed in Nature**) AKA Mixed Effects.*
- *(**Factors are Crossed**) IJ groups – one per (i,j)-level factor combination.*
- *(**Balanced Replication in Groups**) Each group has $K > 1$ units.*
- *(**Distinct Exp. Units**) All IJK units are distinct from each other.*

- *(**Random Assignment across Groups**)*

- *(**Independence**) All measurements on units are independent.*
- *(**Normality**) All groups are approximately normally distributed.*
- *(**Equal Variances**) All groups have approximately same variance.*

Mnemonic: ***2DF FLaMiN FaC BRiG DEU*** | ***RAaG*** | ***I.N.EV***

2F bcrANOVA Linear Model (Mixed Effects)

Mixed effects means one fixed factor and one random factor.

2F bcrANOVA Mixed Effects Linear Model

(I,J)	$\equiv$	(# levels of factor A, # levels of factor B)
K	$\equiv$	# observations (replications) at each (i,j)-level of factors A & B
X_{ijk}	$\equiv$	rv for k^{th} observation at (i,j)-level of factors A & B
μ	$\equiv$	Mean average response over all levels of factors A & B
(A_i, α_j^B)	$\equiv$	(rv for effect of i^{th}-level factor A, effect of j^{th}-level factor B)
AB_{ij}	$\equiv$	rv for interaction between (i,j)-level factors A & B
E_{ijk}	$\equiv$	Deviation from μ due to random error

ASSUMPTIONS:

$E_{ijk} \overset{iid}{\sim}$ Normal $\left(0, \sigma^2\right)$, $A_i \overset{iid}{\sim}$ N $\left(0, \sigma_A^2\right)$, $AB_{ij} \overset{iid}{\sim}$ N $\left(0, \sigma_{AB}^2\right)$

$A_i,\ AB_{ij}\ \&\ E_{ijk}$ are all mutually independent of each other

$$X_{ijk} = \mu + A_i + \alpha_j^B + AB_{ij} + E_{ijk}$$

$H_0^A: \quad \sigma_A^2 = 0$	$H_0^B:$ All $\alpha_j^B = 0$	$H_0^{AB}: \quad \sigma_{AB}^2 = 0$
$H_A^A: \quad \sigma_A^2 > 0$	$H_A^B:$ Some $\alpha_j^B \neq 0$	$H_A^{AB}: \quad \sigma_{AB}^2 > 0$

2F bcrANOVA Mixed Effects F-Test

1. $\nu_A = I - 1,\ \nu_B = J - 1,\ \nu_{AB} = (I-1)(J-1),\ \nu_{res} = IJ(K-1)$
2. $\bar{x}_{i\bullet\bullet} := \frac{1}{JK}\sum_j \sum_k x_{ijk},\quad \bar{x}_{\bullet j\bullet} := \frac{1}{IK}\sum_i \sum_k x_{ijk},\quad \bar{x}_{ij\bullet} := \frac{1}{K}\sum_k x_{ijk}$
3. $\bar{x}_{\bullet\bullet\bullet} := \frac{1}{IJK}\sum_i \sum_j \sum_k x_{ijk}$
4. $$\begin{cases} \text{SS}_{res} & := & \sum_{ijk}(x_{ijk}^{res})^2 & = & \sum_i \sum_j \sum_k (x_{ijk} - \bar{x}_{ij\bullet})^2 \\ \text{SS}_A & := & \sum_{ijk}(\hat{\alpha}_i^A)^2 & = & \sum_i \sum_j \sum_k (\bar{x}_{i\bullet\bullet} - \bar{x}_{\bullet\bullet\bullet})^2 \\ \text{SS}_B & := & \sum_{ijk}(\hat{\alpha}_j^B)^2 & = & \sum_i \sum_j \sum_k (\bar{x}_{\bullet j\bullet} - \bar{x}_{\bullet\bullet\bullet})^2 \\ \text{SS}_{AB} & := & \sum_{ijk}(\hat{\gamma}_{ij}^{AB})^2 & = & \sum_i \sum_j \sum_k (\bar{x}_{ij\bullet} - \bar{x}_{i\bullet\bullet} - \bar{x}_{\bullet j\bullet} + \bar{x}_{\bullet\bullet\bullet})^2 \end{cases}$$

(Optional) $\text{SS}_{total} := \sum_{ijk}(x_{ijk} - \hat{\mu})^2 = \sum_i \sum_j \sum_k (x_{ijk} - \bar{x}_{\bullet\bullet\bullet})^2$

2F bcrANOVA Mixed Effects F-Test

5. $$\text{MS}_A = \frac{\text{SS}_A}{\nu_A}, \quad \text{MS}_B = \frac{\text{SS}_B}{\nu_B}, \quad \text{MS}_{AB} = \frac{\text{SS}_{AB}}{\nu_{AB}}, \quad \text{MS}_{res} = \frac{\text{SS}_{res}}{\nu_{res}}$$

6. $$f_A = \frac{\text{MS}_A}{\text{MS}_{AB}}, \quad f_B = \frac{\text{MS}_B}{\text{MS}_{AB}}, \quad f_{AB} = \frac{\text{MS}_{AB}}{\text{MS}_{res}}$$

7. (if using software): $$\begin{cases} p_A := \mathbb{P}(F > f_A) \approx 1 - \Phi_F(f_A; \nu_A, \nu_{res}) \\ p_B := \mathbb{P}(F > f_B) \approx 1 - \Phi_F(f_B; \nu_B, \nu_{res}) \\ p_{AB} := \mathbb{P}(F > f_{AB}) \approx 1 - \Phi_F(f_{AB}; \nu_{AB}, \nu_{res}) \end{cases}$$

8. $$\begin{cases} \text{If } p_A \leq \alpha \text{ or } f_A > f^*_{\nu_A, \nu_{res}; \alpha} \text{ then reject } H_0^A \text{ else accept } H_0^A \\ \text{If } p_B \leq \alpha \text{ or } f_B > f^*_{\nu_B, \nu_{res}; \alpha} \text{ then reject } H_0^B \text{ else accept } H_0^B \\ \text{If } p_{AB} \leq \alpha \text{ or } f_{AB} > f^*_{\nu_{AB}, \nu_{res}; \alpha} \text{ then reject } H_0^{AB} \text{ else accept } H_0^{AB} \end{cases}$$

2F bcrANOVA Mixed Effects (Expected Mean Squares)

Proposition

Given 2F experiment satisfying 2F bcrANOVA mixed effects assumptions. Then:

$$
\begin{aligned}
(i) \quad \mathbb{E}[MS_{res}] &= \sigma^2 \\
(ii) \quad \mathbb{E}[MS_A] &= \sigma^2 + K\sigma_{AB}^2 + JK\sigma_A^2 \\
(iii) \quad \mathbb{E}[MS_B] &= \sigma^2 + K\sigma_{AB}^2 + \frac{IK}{J-1}\sum_j (\alpha_j^B)^2 \\
(iv) \quad \mathbb{E}[MS_{AB}] &= \sigma^2 + K\sigma_{AB}^2
\end{aligned}
$$

2F bcrANOVA Mixed Effects (Point Estimators of σ^2, σ^2_A & σ^2_{AB})

Proposition

Given 2F experiment satisfying 2F bcrANOVA mixed effects assumptions. Then:

$$
\begin{aligned}
(i) \quad & \hat{\sigma}^2 & = \quad & MS_{res} \\
(ii) \quad & \hat{\sigma}^2_A & = \quad & (MS_A - MS_{AB})/JK \\
(iii) \quad & \hat{\sigma}^2_{AB} & = \quad & (MS_{AB} - MS_{res})/K
\end{aligned}
$$

Fin

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