1F Repeated Measures ANOVA (1F rmANOVA)

Engineering Statistics II

Josh Engwer

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1-Factor Repeated Measures ANOVA (1F rmANOVA)

Recall that 1-Factor ANOVA is a natural extension of independent t-tests.

Moreover, recall paired t-tests (§9.3) where the pairing has, for example:

- each subject being measured pre-treatment and then post-treatment, or
- each subject being measured at two different times.

A **1-Factor Repeated measures ANOVA (1F rmANOVA)** is a natural extension of paired t-tests:

Definition

(Repeated Measures Design)

An experimental design where each subject is measured after taking each of three or more treatments or is measured at three or more different times is called a **repeated measures design**.

NOTE: 1F rmANOVA is essentially a 2F rcbANOVA except:

- The non-blocking factor levels are subjects to be repeatedly measured.
- The blocks are always points in time.
- Each subject is its own block$^{\dagger\ddagger\natural\clubsuit\diamondsuit}$ throughout experiment.

1F rmANOVA (Assumptions)

The 1F rmANOVA assumptions include those of 1F ANOVA and more$^{\dagger\ddagger\natural}$:

Proposition

(1F rmANOVA Assumptions)

1F rmANOVA assumptions include all the 1F ANOVA assumptions...

- ⋆ *(**Randomization**) All measurements are randomly selected.*
- ⋆ *(**Independence**) All measurements are independent.*
- ⋆ *(**Normality**) All populations are approximately normally distributed.*
- ⋆ *(**Same Spread**) All populations have approximately same variance.*

...plus one of the following additional assumptions:

- ⋆ *(**Uniformity**) All populations have equal variances & equal covariances.*
- ⋆ *(**Sphericity**) All pairwise measurement differences have equal variances.*

Sphericity is necessary & sufficient – uniformity is stronger than sphericity.

NOTE: Uniformity assumption is sometimes called **compound symmetry**.

Repeated Measures Designs versus Completely Randomized Designs

Advantages of repeated measures designs$^{\dagger\ddagger\natural\clubsuit\diamondsuit}$:

- Every subject is its own block, removing <u>all</u> variation between subjects due to individuality. ([LH §5.4.1]: "Subjects serve as their own controls..."†)
- Much more economical with respect to necessary subjects.
    - e.g. A completely randomized design with, say, 8 treatments would require, say, 10 subjects per treatment; hence, 80 subjects total. The corresponding repeated measures design would only require 10 subjects total!

Repeated Measures Designs versus Completely Randomized Designs

Disadvantages of repeated measures designs[†‡♠♣]:

- Treatment sequence may undesirably affect subject performance.
  - (**Practice**[♠♣]): Subjects correctly guess next treatment.
  - (**Attrition**[♠♣]): Subjects may leave experiment due to adverse effects.
  - (**Fatigue**[♠♣]): Subjects' performances steadily decline over time.
  - Mitigation: **Counterbalance**[♠♣] the experiment.
- (**Carryover**[♠♣]) Previous treatment's effect persists during current one.
  - Mitigation: Increase time between treatments so that prior effect(s) subside.
- (**Latency**[♣]): One treatment's effect may not be obvious until the introduction of another treatment.
  - Mitigation: (very difficult)[♣]

1F rmANOVA Mixed Effects Linear Model$^{\clubsuit\diamondsuit}$

(I,J)	$\equiv$	(# participants/subjects, # levels of factor A)
X_{ij}	$\equiv$	rv for observation at i^{th} subject & j^{th} level of factor A
μ	$\equiv$	Mean avg response over all subjects and levels of factor A
(P_i, α_j^A)	$\equiv$	(rv for effect of i^{th} subject, Effect of j^{th}-level factor A)
PA_{ij}	$\equiv$	rv for interaction between i^{th} subject & j^{th} level of factor A
E_{ij}	$\equiv$	Deviation from μ due to random error for i^{th} subject in j^{th} level

ASSUMPTIONS:

$E_{ij} \overset{iid}{\sim}$ Normal $\left(0, \sigma^2\right)$, $P_i \overset{iid}{\sim}$ N $\left(0, \sigma_P^2\right)$, $PA_{ij} \overset{iid}{\sim}$ N $\left(0, \sigma_{PA}^2\right)$

P_i & E_{ij} are all mutually independent of each other

PA_{ij} & E_{ij} are all mutually independent of each other

$P_{i'}$ & PA_{ij} are all mutually independent of each other only if $i' \neq i$

$$X_{ij} = \mu + P_i + \alpha_j^A + PA_{ij} + E_{ij} \quad \text{where} \quad \sum_j \alpha_j^A = 0$$

$$H_0^A : \quad \text{All} \quad \alpha_j^A = 0$$
$$H_A^A : \quad \text{Some} \quad \alpha_j^A \neq 0$$

Sums of Squares as a "Partitioning" of Variation Explanation for 1F rmANOVA♣

$$\underbrace{\text{SS}_{total}}_{\substack{\textit{Total Variation}\\ \textit{in Experiment}}} = \underbrace{\text{SS}_{P}}_{\substack{\textit{Variation due}\\ \textit{to Subject}\\ (\textit{between subjects})}} + \underbrace{\text{SS}_{A}}_{\substack{\textit{Variation due}\\ \textit{to Factor A}\\ (\textit{within subjects})}} + \underbrace{\text{SS}_{PA}}_{\substack{\textit{Variation due}\\ \textit{to Interaction}\\ (\textit{within subjects})}}$$

$$\underbrace{\nu}_{\substack{\textit{Total dof's in}\\ \textit{Experiment}}} = \underbrace{\nu_{P}}_{\substack{\textit{Subject}\\ \textit{dof's}\\ (\textit{between subjects})}} + \underbrace{\nu_{A}}_{\substack{\textit{Factor A}\\ \textit{dof's}\\ (\textit{within subjects})}} + \underbrace{\nu_{PA}}_{\substack{\textit{Interaction}\\ \textit{dof's}\\ (\textit{within subjects})}}$$

$$\nu = IJ - 1, \qquad \nu_P = I - 1, \qquad \nu_A = J - 1, \qquad \nu_{PA} = (I-1)(J-1)$$

Because each subject-treatment group has only one measurement, the interaction and residual are **confounded**.
In other words, it's impossible to determine how much of SS_{PA} is due to the interaction and how much is due to random error/residual.♣

1F rmANOVA F-Test$^{\clubsuit\natural}$

1. $\nu_P = I - 1, \quad \nu_A = J - 1, \quad \nu_{PA} = (I-1)(J-1)$
2. $\bar{x}_{i\bullet} := \frac{1}{J}\sum_j x_{ij}, \qquad \bar{x}_{\bullet j} := \frac{1}{I}\sum_i x_{ij}, \qquad \bar{x}_{\bullet\bullet} := \frac{1}{IJ}\sum_i\sum_j x_{ij}$
3. $\begin{cases} \text{SS}_{total} & = & \sum_i\sum_j (x_{ij} - \bar{x}_{\bullet\bullet})^2 \\ \text{SS}_P & = & \sum_i\sum_j (\bar{x}_{i\bullet} - \bar{x}_{\bullet\bullet})^2 \\ \text{SS}_A & = & \sum_i\sum_j (\bar{x}_{\bullet j} - \bar{x}_{\bullet\bullet})^2 \\ \text{SS}_{PA} & := & \text{SS}_{total} - \text{SS}_P - \text{SS}_A \end{cases}$
4. $\text{MS}_P = \dfrac{\text{SS}_P}{\nu_P}, \quad \text{MS}_A = \dfrac{\text{SS}_A}{\nu_A}, \quad \text{MS}_{PA} = \dfrac{\text{SS}_{PA}}{\nu_{PA}} \implies f_A = \dfrac{\text{MS}_A}{\text{MS}_{PA}}$
5. (if using software): $\quad p_A \; := \; \mathbb{P}(F > f_A) \; \approx \; 1 - \Phi_F(f_A; \nu_A, \nu_{PA})$
6. If $\; p_A \le \alpha \;$ or $\; f_A > f^*_{\nu_A,\nu_{PA};\alpha} \;$ then reject $\; H_0^A \;$ else accept $\; H_0^A$

1F rmANOVA (Table)

$$\nu_P := I-1, \quad \nu_A := J-1, \quad \nu_{PA} := (I-1)(J-1), \quad \nu := IJ-1$$

1F rmANOVA Table$^\natural$ (**Significance Level** α)					
Variation Source	**df**	**Sum of Squares**	**Mean Square**	F **Stat Value***	**Decision**
Between Subjects:					
P	ν_P	SS_P	MS_P		
Within Subjects:					
A	ν_A	SS_A	MS_A	f_A	Acc/Rej H_0^A
PA	ν_{PA}	SS_{PA}	MS_{PA}		
Total	ν	SS_{total}			

* For violated sphericity, use **adjusted Huynh-Feldt** F**-test** instead$^{\dagger\natural}$:

H. Huynh, L.S. Feldt, “Conditions under which Mean Square Ratios in Repeated Measurement Designs have exact F-distributions”, *J. American Statistical Association*, **65** (1970), 1582-1589.

1F rmANOVA (Expected Mean Squares)

Proposition

Given experiment satisfying 1F rmANOVA assumptions. Then:

$$
\begin{aligned}
(i) \quad & \mathbb{E}[MS_P] &=& \quad \sigma^2 + J\sigma_P^2 \\
(iI) \quad & \mathbb{E}[MS_A] &=& \quad \sigma^2 + \sigma_{PA}^2 + \frac{I}{J-1}\sum_j (\alpha_j^A)^2 \\
(iii) \quad & \mathbb{E}[MS_{PA}] &=& \quad \sigma^2 + \sigma_{PA}^2
\end{aligned}
$$

1F rmANOVA (Post-Hoc Comparisons)

Regarding post-hoc comparisons for rmANOVA:

- If compound symmetry assumption is satisfied,
  then Tukey Post-Hoc comparisons can be used accordingly[†‡♮]:

  G. Keppel, T.D. Wickens, *Design and Analysis: A Researcher's Handbook*, 3^{rd} Ed., Pearson, 2004.

- If compound symmetry assumption is violated, then multiple paired t-tests with Bonferroni-corrected significance levels are recommended[†]:

  S.E. Maxwell, "Pairwise Multiple Comparisons in Repeated Measures Designs", *J. Educational Statistics*, **5** (1980), 269-287.

2F rmANOVA & 3F rmANOVA

2-Factor Repeated Measures ANOVA (2F rmANOVA) and higher are beyond the scope of this course.

Consult the following for more info on 1F rmANOVA:

†R.G. Lomax, D.L. Hahs-Vaughn, *Statistical Concepts: A* 2^{nd} *Course*, 4^{th} Ed, Routledge, 2012.

‡J.P. Stevens, *Intermediate Statistics: A Modern Approach*, 3^{rd} Ed, Taylor & Francis, 2007.

♮W.L. Hays, *Statistics*, 5^{th} Ed, Harcourt Brace & Company, 1994. (§13.21, §13.22, §13.25)

Consult the following for more info on 1F/2F/3F rmANOVA:

♠C.P. Doncaster, A. Davey, *Analysis of Variance and Covariance*, Cambridge Press, 2007. (Ch6)

♣E.R. Girden, *ANOVA: Repeated Measures*, SAGE, 1992.

♢B.J. Winer, *Statistical Principles in Experimental Design*, 1^{st} Ed, McGraw-Hill, 1962. (Ch4, Ch7)

Fin.