Independent & Pooled t-Tests/t-CI's for $\mu_1 - \mu_2$

Engineering Statistics II
Section 9.2

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2018

PART I:

Gosset's t Distribution

William Sealy Gosset (1876-1937)

Gosset's employer made him publish under the pseudonym "Student". Hence, some textbooks/papers use the term "Student's t Distribution".

Gosset's t Distribution

Definition

Notation	$T \sim t_\nu$
Parameters	$\nu \equiv$ # Degrees of Freedom $\quad (\nu = 1, 2, 3, \cdots)$
Support	Supp$(T) = (-\infty, \infty)$
pdf	$f_T(t;\nu) := \frac{\Gamma((\nu+1)/2)}{\sqrt{\pi\nu}\cdot\Gamma(\nu/2)} \cdot \frac{1}{[1+(t^2/\nu)]^{(\nu+1)/2}}$
cdf	$\Phi_t(t;\nu) = \frac{\Gamma((\nu+1)/2)}{\sqrt{\pi\nu}\cdot\Gamma(\nu/2)} \int_{-\infty}^{t} \frac{1}{[1+(\tau^2/\nu)]^{(\nu+1)/2}}\, d\tau$
Mean	$\mathbb{E}[T] = +\infty$, for $\nu = 1$ $\mathbb{E}[T] = 0$, for $\nu > 1$
Variance	$\mathbb{V}[T] = +\infty$, for $\nu = 1, 2$ $\mathbb{V}[T] = \nu/(\nu - 2)$, for $\nu > 2$
Model(s)	(Used exclusively for Statistical Inference)

ν is the lowercase Greek letter "nu" $\quad$ τ is the lowercase Greek letter "tau"

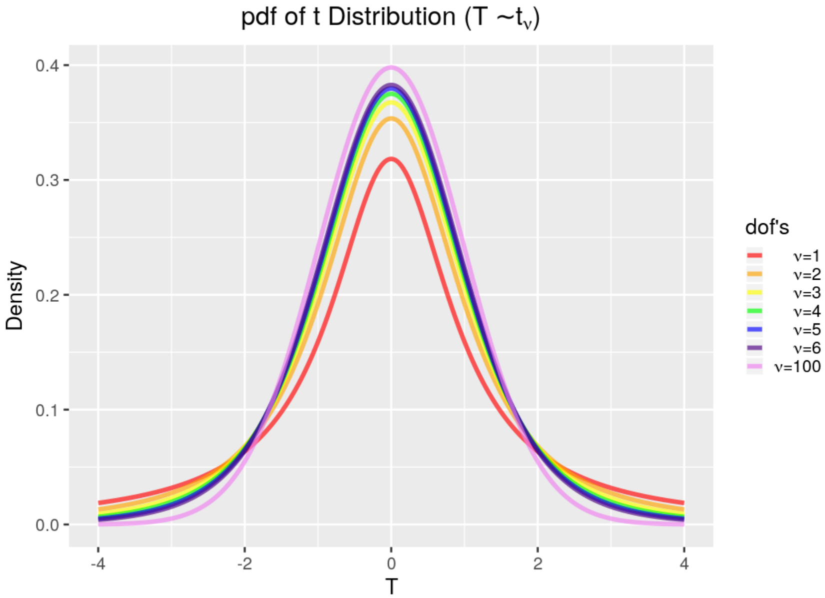
Gosset's t Distribution (Properties)

Proposition

Properties of t distributions:

- *The t_ν pdf curve is symmetric, bell-shaped and centered at zero.*
- *The t_ν pdf curve is more spread out than the std normal pdf curve.*
- *The spread of the t_ν pdf curve decreases as ν increases.*
- *As $\nu \to \infty$, the t_ν pdf curves approaches the std normal pdf curve.*
- *Let independent rv's* $\begin{cases} Z & \sim & \text{StdNormal} \\ X & \sim & \chi^2_\nu \end{cases}$. *Then* $\dfrac{Z}{\sqrt{X/\nu}} \sim t_\nu$

PROOF: Beyond scope of course. Take **Mathematical Statistics**.


pdf of t Distribution (T ~t$_\nu$)
Density
0.4
0.3
0.2
0.1
0.0
-4
-2
0
2
4
T
dof's
ν=1
ν=2
ν=3
ν=4
ν=5
ν=6
ν=100

t-Cutoffs (AKA t Critical Values) (Definition)

A key component to some CI's & hypothesis tests is the t**-cutoff**:

Definition

$t^*_{\nu;\alpha}$ is called a t**-cutoff** of the t_ν distribution such that its upper-tail probability is exactly its subscript value α: (Here, $T \sim t_\nu$)

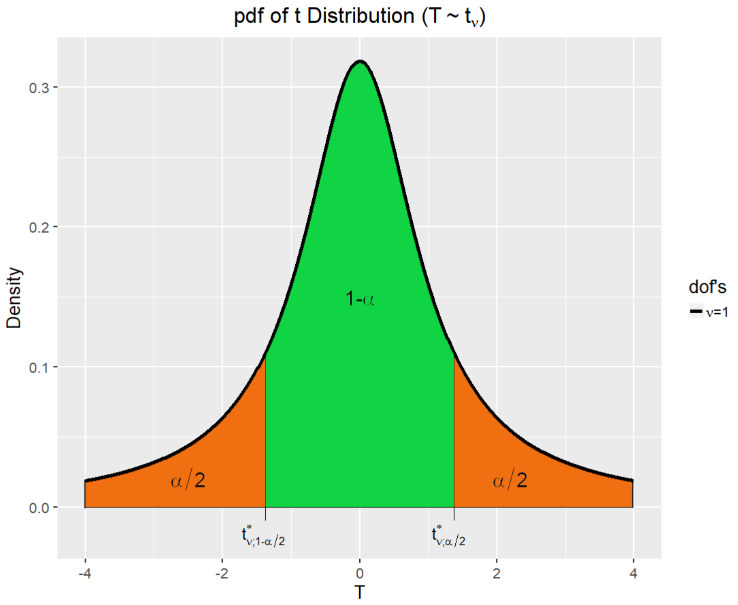
$$\mathbb{P}(T > t^*_{\nu;\alpha}) = \alpha$$

<u>NOTE:</u> Do <u>not</u> confuse t-cutoff $t^*_{\nu;\alpha}$ with t percentile $t_{\nu;\alpha}$:

$$\mathbb{P}(T \le t_{\nu;\alpha}) = \alpha$$

Another name for t-cutoff is t **critical value**.

t-Cutoffs (Example Plot)



t-Cutoffs (Key Property)

Proposition

Lower-tail t-cutoffs can be determined from appropriate upper-tail t-cutoffs:

$$t^*_{\nu;1-\alpha} = -t^*_{\nu;\alpha}$$

PROOF: Follows from the fact that t distributions are symmetric.

t-Cutoffs Table

GOSSET'S t-CUTOFFS, $t^*_{\nu;\alpha}$ $\quad \mathbb{P}(T > t^*_{\nu;\alpha}) = \alpha, \quad t^*_{\nu;1-\alpha} = -t^*_{\nu;\alpha}$

ν \ α	0.2	0.1	0.05	0.025	0.02	0.01	0.005	0.001	0.0005
1	1.376	3.078	6.314	12.706	15.895	31.821	63.657	318.309	636.619
2	1.061	1.886	2.920	4.303	4.849	6.965	9.925	22.327	31.599
3	0.978	1.638	2.353	3.182	3.482	4.541	5.841	10.215	12.924
4	0.941	1.533	2.132	2.776	2.999	3.747	4.604	7.173	8.610
5	0.920	1.476	2.015	2.571	2.757	3.365	4.032	5.893	6.869
6	0.906	1.440	1.943	2.447	2.612	3.143	3.707	5.208	5.959
7	0.896	1.415	1.895	2.365	2.517	2.998	3.499	4.785	5.408
8	0.889	1.397	1.860	2.306	2.449	2.896	3.355	4.501	5.041
9	0.883	1.383	1.833	2.262	2.398	2.821	3.250	4.297	4.781
10	0.879	1.372	1.812	2.228	2.359	2.764	3.169	4.144	4.587
11	0.876	1.363	1.796	2.201	2.328	2.718	3.106	4.025	4.437
12	0.873	1.356	1.782	2.179	2.303	2.681	3.055	3.930	4.318
13	0.870	1.350	1.771	2.160	2.282	2.650	3.012	3.852	4.221
14	0.868	1.345	1.761	2.145	2.264	2.624	2.977	3.787	4.140
15	0.866	1.341	1.753	2.131	2.249	2.602	2.947	3.733	4.073
16	0.865	1.337	1.746	2.120	2.235	2.583	2.921	3.686	4.015
17	0.863	1.333	1.740	2.110	2.224	2.567	2.898	3.646	3.965

PART II:

Independent t-Tests & Independent t-CI's (Unknown Population Variances σ_1^2, σ_2^2)

A Statistic Related to the t Distribution

Theorem

Let $\mathbf{X} := (X_1, \ldots, X_{n_1})$ *be a random sample from a* Normal(μ_1, σ_1^2) *population.*
Let $\mathbf{Y} := (Y_1, \ldots, Y_{n_2})$ *be a random sample from a* Normal(μ_2, σ_2^2) *population.*
Moreover, suppose random samples $\mathbf{X}$ & $\mathbf{Y}$ *are* independent *of each other.*

Then: $$\frac{(\overline{X} - \overline{Y}) - (\mu_1 - \mu_2)}{\sqrt{\dfrac{S_1^2}{n_1} + \dfrac{S_2^2}{n_2}}} \overset{approx}{\sim} t_{\nu^*} \quad \text{where} \quad \nu^* = \left\lfloor \frac{\left(\dfrac{s_1^2}{n_1} + \dfrac{s_2^2}{n_2}\right)^2}{\dfrac{(s_1^2/n_1)^2}{n_1 - 1} + \dfrac{(s_2^2/n_2)^2}{n_2 - 1}} \right\rfloor$$

PROOF: Beyond scope of course. Take **Mathematical Statistics**.

Independent t-Test for $\mu_1 - \mu_2$ (Unknown σ_1, σ_2)

Proposition

Population:	*Two Normal Populations with unknown* σ_1, σ_2
Realized Samples:	$\mathbf{x} := (x_1, x_2, \cdots, x_{n_1})$ *with mean* $\bar{x}$, *std dev* s_1 $\mathbf{y} := (y_1, y_2, \cdots, y_{n_2})$ *with mean* $\bar{y}$, *std dev* s_2 *Samples* $\mathbf{x}$ & $\mathbf{y}$ *are independent of each other*
Test Statistic Value: $W(\mathbf{x}, \mathbf{y}; \delta_0)$	$t = \dfrac{(\bar{x} - \bar{y}) - \delta_0}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}}, \quad \nu^* = \left\lfloor \dfrac{\left(\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}\right)^2}{\frac{(s_1^2/n_1)^2}{n_1 - 1} + \frac{(s_2^2/n_2)^2}{n_2 - 1}} \right\rfloor$

HYPOTHESIS TEST:	***REJECTION REGION AT LVL*** α :
$H_0 : \mu_1 - \mu_2 = \delta_0$ vs. $H_A : \mu_1 - \mu_2 > \delta_0$	$t \geq t^*_{\nu^*;\alpha}$
$H_0 : \mu_1 - \mu_2 = \delta_0$ vs. $H_A : \mu_1 - \mu_2 < \delta_0$	$t \leq t^*_{\nu^*;1-\alpha}$
$H_0 : \mu_1 - \mu_2 = \delta_0$ vs. $H_A : \mu_1 - \mu_2 \neq \delta_0$	$t \leq t^*_{\nu^*;1-\alpha/2}$ or $t \geq t^*_{\nu^*;\alpha/2}$

Independent t-Test for $\mu_1 - \mu_2$ (Unknown σ_1, σ_2)

Proposition

Population:	*Two Normal Populations with unknown* σ_1, σ_2
Realized Samples:	$\mathbf{x} := (x_1, x_2, \cdots, x_{n_1})$ *with mean* $\bar{x}$, *std dev* s_1 $\mathbf{y} := (y_1, y_2, \cdots, y_{n_2})$ *with mean* $\bar{y}$, *std dev* s_2 *Samples* $\mathbf{x}$ & $\mathbf{y}$ *are independent of each other*
Test Statistic Value: $W(\mathbf{x}, \mathbf{y}; \delta_0)$	$t = \dfrac{(\bar{x} - \bar{y}) - \delta_0}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}}, \quad \nu^* = \left\lfloor \dfrac{\left(\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}\right)^2}{\frac{(s_1^2/n_1)^2}{n_1 - 1} + \frac{(s_2^2/n_2)^2}{n_2 - 1}} \right\rfloor$

HYPOTHESIS TEST:	***P-VALUE DETERMINATION:***
$H_0 : \mu_1 - \mu_2 = \delta_0$ *vs.* $H_A : \mu_1 - \mu_2 > \delta_0$	*P-value* $= 1 - \Phi_t(t; \nu^*)$
$H_0 : \mu_1 - \mu_2 = \delta_0$ *vs.* $H_A : \mu_1 - \mu_2 < \delta_0$	*P-value* $= \Phi_t(t; \nu^*)$
$H_0 : \mu_1 - \mu_2 = \delta_0$ *vs.* $H_A : \mu_1 - \mu_2 \neq \delta_0$	*P-value* $= 2 \cdot [1 - \Phi_t(\lvert t \rvert; \nu^*)]$

DECISION RULE:
If P-value $\leq \alpha$ *then reject* H_0 *in favor of* H_A
If P-value $> \alpha$ *then accept* H_0 (*i.e. fail to reject* H_0)

t-CI for Normal Pop. Difference $\mu_1 - \mu_2$ (Motivation)

Let $\mathbf{X} := (X_1, \ldots, X_{n_1})$ be a random sample from a Normal(μ_1, σ_1^2) population.
Let $\mathbf{Y} := (Y_1, \ldots, Y_{n_2})$ be a random sample from a Normal(μ_2, σ_2^2) population.
Moreover, suppose random samples $\mathbf{X}$ & $\mathbf{Y}$ are independent of each other.
Then, construct the $100(1-\alpha)\%$ CI for parameter difference $\mu_1 - \mu_2$:

1. Produce suitable **pivot**: $Q(\mathbf{X}, \mathbf{Y}; \mu_1, \mu_2) := [(\overline{X} - \overline{Y}) - (\mu_1 - \mu_2)] / \sqrt{\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}}$
2. Then the pivot is approximately an t distribution: $Q(\mathbf{X}, \mathbf{Y}; \mu_1, \mu_2) \overset{approx}{\sim} t_{\nu^*}$
3. Find constants $a < b$ such that $\mathbb{P}(a < Q(\mathbf{X}, \mathbf{Y}; \mu_1, \mu_2) < b) = 1 - \alpha$

   Since t_ν pdf is symmetric, $\begin{cases} a = t^*_{\nu^*;1-\alpha/2} = -t^*_{\nu^*;\alpha/2} \\ b = t^*_{\nu^*;\alpha/2} \end{cases}$
4. Manipulate the inequalities to isolate parameter difference $\mu_1 - \mu_2$:
$$(\overline{X} - \overline{Y}) - t^*_{\nu^*;\alpha/2} \cdot \sqrt{\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}} < \mu_1 - \mu_2 < (\overline{X} - \overline{Y}) + t^*_{\nu^*;\alpha/2} \cdot \sqrt{\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}}$$
5. Take independent samples $\mathbf{x} := (x_1, \cdots, x_{n_1})$ & $\mathbf{y} := (y_1, \cdots, y_{n_2})$.
6. Replace point estimators $\overline{X}, \overline{Y}, S_1, S_2$ with $\bar{x}, \bar{y}, s_1, s_2$ from the samples:
$$(\bar{x} - \bar{y}) - t^*_{\nu^*;\alpha/2} \cdot \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}} < \mu_1 - \mu_2 < (\bar{x} - \bar{y}) + t^*_{\nu^*;\alpha/2} \cdot \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}$$

Independent t-CI for $\mu_1 - \mu_2$ (Unknown σ_1, σ_2)

Proposition

Given two normal populations with means μ_1 and μ_2.
Let $x_1, x_2, \cdots, x_{n_1}$ be a sample taken from 1^{st} population.
Let $y_1, y_2, \cdots, y_{n_2}$ be a sample taken from 2^{nd} population.
Moreover, suppose samples $\mathbf{x}$ & $\mathbf{y}$ are independent of each other.

Then the $100(1-\alpha)\%$ ***independent t-CI for $\mu_1 - \mu_2$*** *is*

$$\left((\overline{x} - \overline{y}) - t^*_{\nu^*;\alpha/2} \cdot \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}, \ (\overline{x} - \overline{y}) + t^*_{\nu^*;\alpha/2} \cdot \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}} \right)$$

—- OR WRITTEN MORE COMPACTLY —-

$$(\overline{x} - \overline{y}) \pm t^*_{\nu^*;\alpha/2} \cdot \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}$$

where $$\nu^* = \left\lfloor \frac{\left(\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2} \right)^2}{\frac{(s_1^2/n_1)^2}{n_1 - 1} + \frac{(s_2^2/n_2)^2}{n_2 - 1}} \right\rfloor$$

PART III:

Pooled t-Tests & Pooled t-CI's
(Unknown Population Variances $\sigma_1^2 = \sigma_2^2$)

A Pooled Statistic Related to the t Distribution

For the case when the two normal populations have equal variance, there is a better statistic to use in t-tests & t-CI's called a **pooled statistic**:

Theorem

Let $\mathbf{X} := (X_1, \ldots, X_{n_1})$ *be a random sample from a* *Normal*(μ_1, σ^2) *population.*
Let $\mathbf{Y} := (Y_1, \ldots, Y_{n_2})$ *be a random sample from a* *Normal*(μ_2, σ^2) *population.*
Moreover, suppose random samples $\mathbf{X}$ & $\mathbf{Y}$ *are independent of each other.*
Then:

$$\frac{(\overline{X} - \overline{Y}) - (\mu_1 - \mu_2)}{\sqrt{S^2_{pool} \cdot \left(\frac{1}{n_1} + \frac{1}{n_2}\right)}} \sim t_{n_1+n_2-2} \quad \text{where} \quad S^2_{pool} := \frac{(n_1 - 1)S_1^2 + (n_2 - 1)S_2^2}{n_1 + n_2 - 2}$$

PROOF: Beyond scope of course. Take **Mathematical Statistics**.

S^2_{pool} is the weighted average of the two sample variances. This means the sample with more data provides more information about the population variance σ^2 and, hence, its sample variance has more weight in the average.

Pooled t-Test for $\mu_1 - \mu_2$ (Unknown $\sigma_1 = \sigma_2$)

Proposition

Population:	*Two Normal Populations with unknown* $\sigma_1 = \sigma_2$
Realized Samples:	$\mathbf{x} := (x_1, x_2, \cdots, x_{n_1})$ *with mean* $\bar{x}$, *std dev* s_1 $\mathbf{y} := (y_1, y_2, \cdots, y_{n_2})$ *with mean* $\bar{y}$, *std dev* s_2 *Samples* $\mathbf{x}$ & $\mathbf{y}$ *are independent of each other*
Pooled Sample Variance	$s_{pool}^2 = \dfrac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2}$
Test Statistic Value: $W(\mathbf{x}, \mathbf{y}; \delta_0)$	$t_{pool} = \dfrac{(\bar{x} - \bar{y}) - \delta_0}{\sqrt{s_{pool}^2 \cdot \left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}, \quad \nu_{pool} = n_1 + n_2 - 2$

HYPOTHESIS TEST:	***REJECTION REGION AT LVL*** α :
$H_0 : \mu_1 - \mu_2 = \delta_0$ *vs.* $H_A : \mu_1 - \mu_2 > \delta_0$	$t_{pool} \geq t^*_{\nu_{pool};\alpha}$
$H_0 : \mu_1 - \mu_2 = \delta_0$ *vs.* $H_A : \mu_1 - \mu_2 < \delta_0$	$t_{pool} \leq t^*_{\nu_{pool};1-\alpha}$
$H_0 : \mu_1 - \mu_2 = \delta_0$ *vs.* $H_A : \mu_1 - \mu_2 \neq \delta_0$	$t_{pool} \leq t^*_{\nu_{pool};1-\alpha/2}$ *or* $t_{pool} \geq t^*_{\nu_{pool};\alpha/2}$

Pooled t-Test for $\mu_1 - \mu_2$ (Unknown $\sigma_1 = \sigma_2$)

Proposition

Population:	*Two Normal Populations with unknown* $\sigma_1 = \sigma_2$
Realized Samples:	$\mathbf{x} := (x_1, x_2, \cdots, x_{n_1})$ *with mean* $\bar{x}$, *std dev* s_1 $\mathbf{y} := (y_1, y_2, \cdots, y_{n_2})$ *with mean* $\bar{y}$, *std dev* s_2 *Samples* $\mathbf{x}$ & $\mathbf{y}$ *are independent of each other*
Pooled Sample Variance	$s^2_{pool} = \frac{(n_1-1)s_1^2+(n_2-1)s_2^2}{n_1+n_2-2}$
Test Statistic Value: $W(\mathbf{x}, \mathbf{y}; \delta_0)$	$t_{pool} = \dfrac{(\bar{x} - \bar{y}) - \delta_0}{\sqrt{s^2_{pool} \cdot \left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}, \quad \nu_{pool} = n_1 + n_2 - 2$

HYPOTHESIS TEST:	***P-VALUE DETERMINATION:***
$H_0 : \mu_1 - \mu_2 = \delta_0$ *vs.* $H_A : \mu_1 - \mu_2 > \delta_0$	*P-value* $= 1 - \Phi_t(t_{pool}; \nu_{pool})$
$H_0 : \mu_1 - \mu_2 = \delta_0$ *vs.* $H_A : \mu_1 - \mu_2 < \delta_0$	*P-value* $= \Phi_t(t_{pool}; \nu_{pool})$
$H_0 : \mu_1 - \mu_2 = \delta_0$ *vs.* $H_A : \mu_1 - \mu_2 \neq \delta_0$	*P-value* $= 2 \cdot [1 - \Phi_t(\lvert t_{pool}\rvert; \nu_{pool})]$

DECISION RULE:
If P-value $\leq \alpha$ *then reject* H_0 *in favor of* H_A
If P-value $> \alpha$ *then accept* H_0 (*i.e. fail to reject* H_0)

Pooled t-CI for $\mu_1 - \mu_2$ (Unknown $\sigma_1 = \sigma_2$)

Proposition

Given two normal populations with means μ_1 and μ_2 and unknown $\sigma_1^2 = \sigma_2^2$.
Let $x_1, x_2, \cdots, x_{n_1}$ be a sample taken from the 1^{st} population.
Let $y_1, y_2, \cdots, y_{n_2}$ be a sample taken from the 2^{nd} population.
Moreover, suppose the samples $\mathbf{x}$ & $\mathbf{y}$ are independent of each other.

Then the $100(1-\alpha)\%$ ***pooled t-CI for $\mu_1 - \mu_2$*** *is*

$$\left((\bar{x} - \bar{y}) - t^*_{\nu_{pool};\alpha/2} \cdot \sqrt{s^2_{pool} \cdot \left(\frac{1}{n_1} + \frac{1}{n_2}\right)},\ (\bar{x} - \bar{y}) + t^*_{\nu_{pool};\alpha/2} \cdot \sqrt{s^2_{pool} \cdot \left(\frac{1}{n_1} + \frac{1}{n_2}\right)} \right)$$

—- OR WRITTEN MORE COMPACTLY —-

$$(\bar{x} - \bar{y}) \pm t^*_{\nu_{pool};\alpha/2} \cdot \sqrt{s^2_{pool} \cdot \left(\frac{1}{n_1} + \frac{1}{n_2}\right)}$$

where $s^2_{pool} = \dfrac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2}$ *and* $\nu_{pool} = n_1 + n_2 - 2$

Independent t-tests/CI's vs. Pooled t-tests/CI's

While a pooled t-test & t-CI bear higher power & narrower width respectively compared to the corresponding independent t-test & t-CI, they are not robust to violations of equality of population variances.

This lack of robustness in pooled t-tests & pooled t-CI's is reflected by the severe errors that occur when the two population variances differ by even a small amount.

Therefore, when unsure if two given normal populations truly have identical variances (and most often they don't), it is recommended to play it safe and use independent t-tests & t-CI's.

Textbook Logistics for Section 9.2

CONCEPT	TEXTBOOK NOTATION	SLIDES/OUTLINE NOTATION
Probability of Event $A \subseteq \Omega$	$P(A)$	$\mathbb{P}(A)$
Expected Value of rv X	$E(X)$	$\mathbb{E}[X]$
Variance of rv X	$V(X)$	$\mathbb{V}[X]$
Alternative Hypothesis	H_a	H_A
Sample Sizes	m, n	n_1, n_2
Pooled t-Test Stat Value	t	t_{pool}
t-Cutoffs	$t_{\alpha,\nu}, \quad t_{\alpha/2,\nu}$	$t^*_{\nu;\alpha}, \quad t^*_{\nu;\alpha/2}$
Hypothesized Mean Difference	Δ_0	δ_0

- Ignore “Type II Error Probabilities” section. (pg 340-341)
  - Turns out computing the power of a 2-sample t-test is complicated.
- Ignore any mention of **one-sided CI’s**.

Fin.