F-Tests & F-CI's for Normal Pop. Variances

Engineering Statistics II
Section 9.5

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TTU

2018

PART I:

Snedecor's F Distribution

George Waddel Snedecor (1881-1974)

Snedecor founded the first statistics department at Iowa State University.

Snedecor's F Distribution

Definition

Notation	$F \sim F_{\nu_1,\nu_2}$
Parameters	$\nu_1 \equiv$ # "Top" dof's $(\nu_1 = 1, 2, 3, \cdots)$ $\nu_2 \equiv$ # "Bottom" dof's $(\nu_2 = 1, 2, 3, \cdots)$
Support	Supp$(F) = (0, \infty)$ if $\nu_1 = 1$; Supp$(F) = [0, \infty)$ otherwise
pdf	$f_F(x; \nu_1, \nu_2) := \frac{(\nu_1/\nu_2)^{\nu_1/2}}{B(\nu_1/2, \nu_2/2)} \cdot x^{(\nu_1/2)-1} \left(1 + \frac{\nu_1}{\nu_2} x\right)^{-(\nu_1+\nu_2)/2}$
cdf	$\Phi_F(x; \nu_1, \nu_2) = \frac{(\nu_1/\nu_2)^{\nu_1/2}}{B(\nu_1/2, \nu_2/2)} \int_0^x \xi^{(\nu_1/2)-1} \left(1 + \frac{\nu_1}{\nu_2} \xi\right)^{-(\nu_1+\nu_2)/2} d\xi$
Mean	$\mathbb{E}[F] = +\infty$, for $\nu_2 = 1, 2$ $\mathbb{E}[F] = \frac{\nu_2}{\nu_2 - 2}$, for $\nu_2 > 2$
Variance	$\mathbb{V}[F] = +\infty$, for $\nu_2 = 1, 2, 3, 4$ $\mathbb{V}[F] = \frac{2\nu_2^2(\nu_1+\nu_2-2)}{\nu_1(\nu_2-2)^2(\nu_2-4)}$, for $\nu_2 > 4$
Model(s)	(Used exclusively for Statistical Inference)

$B(\cdot, \cdot) \equiv$ Beta Function

ξ is the lowercase Greek letter "xi"

Snedecor named it the F distribution in honor of statistician Sir Ronald Fisher.

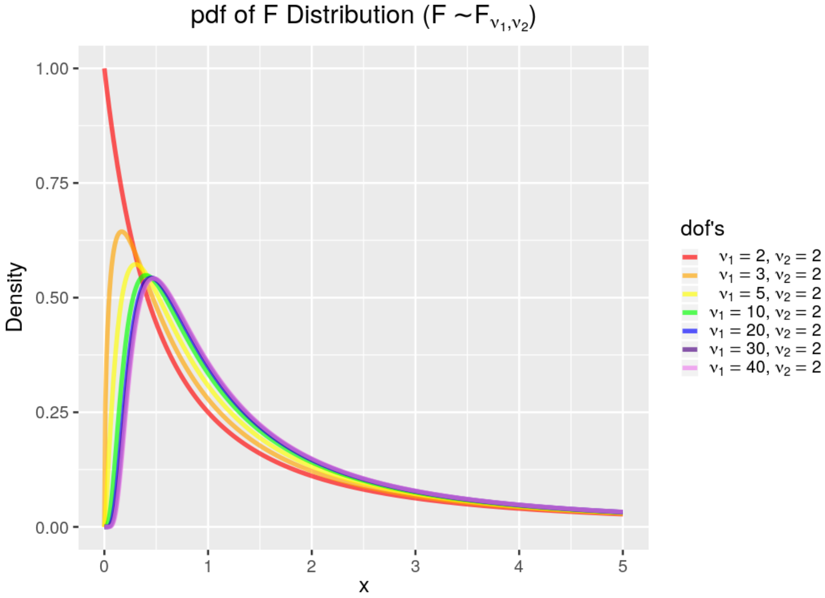
Proposition

Properties of F distributions:

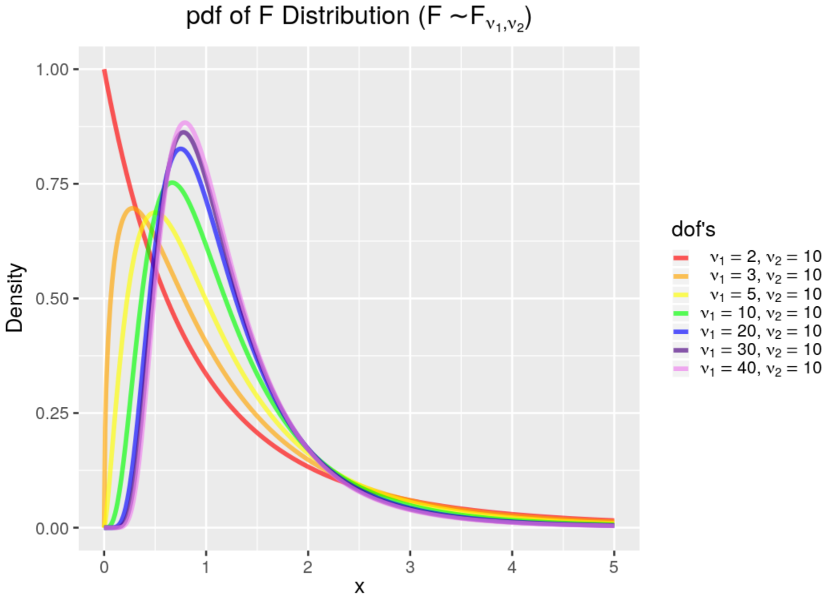
- *The F_{ν_1,ν_2} pdf curve is positively skewed.*
- *Let random variable* $X \sim F_{\nu_1,\nu_2}$. *Then* $\dfrac{1}{X} \sim F_{\nu_2,\nu_1}$
- *Let random variable* $X \sim t_n$. *Then* $X^2 \sim F_{1,n}$
- *Let* X_1, X_2 *be independent rv's such that* $X_1 \sim \chi^2_{\nu_1}$ & $X_2 \sim \chi^2_{\nu_2}$. *Then* $\dfrac{X_1/\nu_1}{X_2/\nu_2} \sim F_{\nu_1,\nu_2}$
- *Let* X_1, X_2 *be independent rv's s.t.* $X_1 \sim$ *Gamma*(α_1, β_1) *and* $X_2 \sim$ *Gamma*(α_2, β_2). *Then* $\dfrac{\alpha_2\beta_1X_1}{\alpha_1\beta_2X_2} \sim F_{2\alpha_1,2\alpha_2}$

PROOF: Beyond scope of course. Take **Mathematical Statistics**.

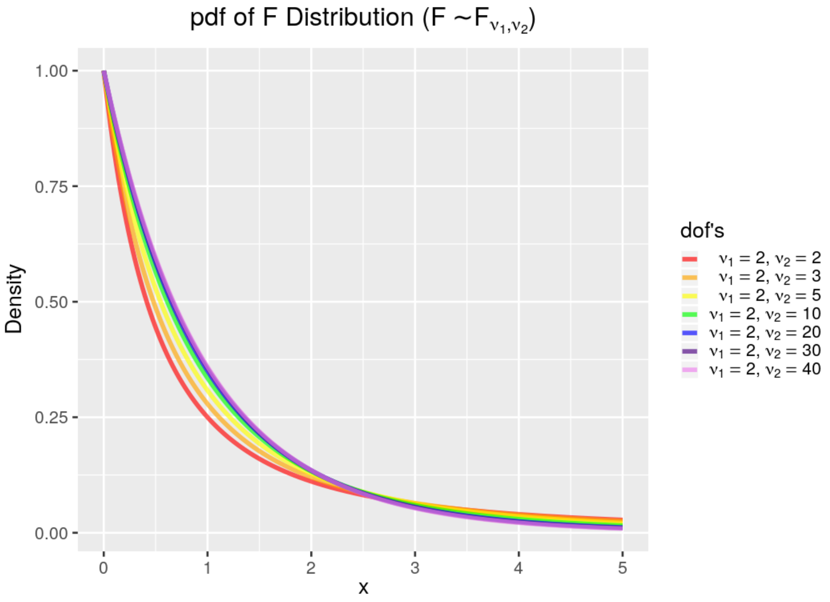
Plot of F Distributions (ν_1 grows & $\nu_2 = 2$ is fixed)



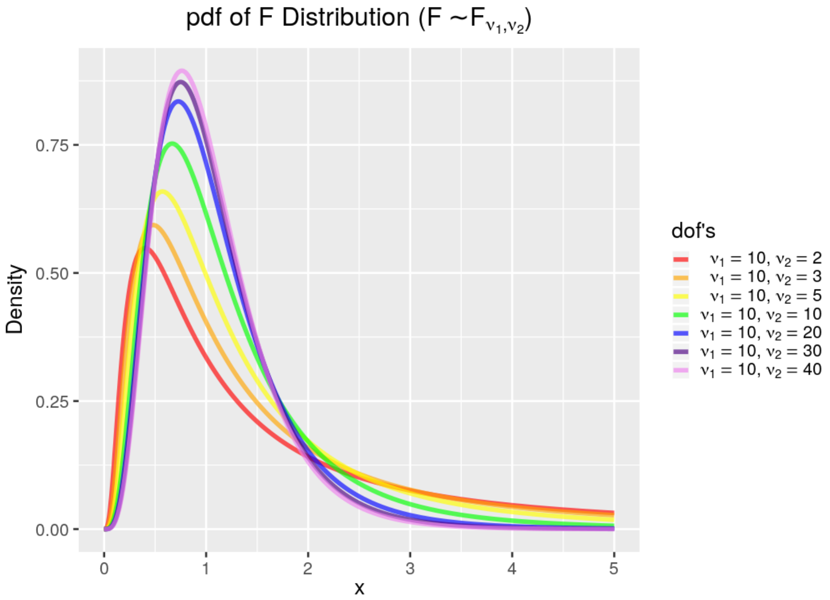
Plot of F Distributions (ν_1 grows & $\nu_2 = 10$ is fixed)



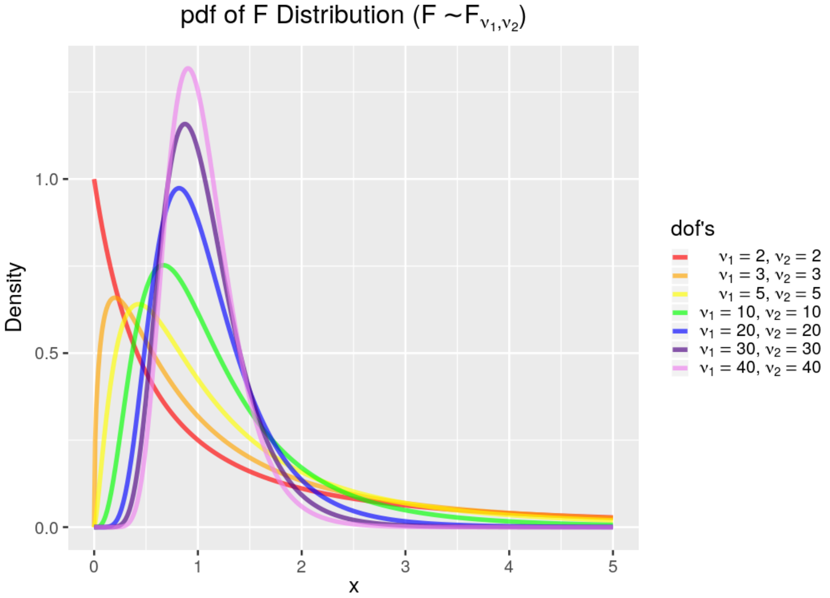
Plot of F Distributions ($\nu_1 = 2$ is fixed & ν_2 grows)



Plot of F Distributions ($\nu_1 = 10$ is fixed & ν_2 grows)



Plot of F Distributions (ν_1 & ν_2 both grow in unison)



F-Cutoffs (AKA F Critical Values) (Definition)

A key component to some CI's & hypothesis tests is the F**-cutoff**:

Definition

$f^*_{\nu_1,\nu_2;\alpha}$ is called a F**-cutoff** of the F_{ν_1,ν_2} distribution such that its upper-tail probability is exactly its subscript value α: (Here, $F \sim F_{\nu_1,\nu_2}$)

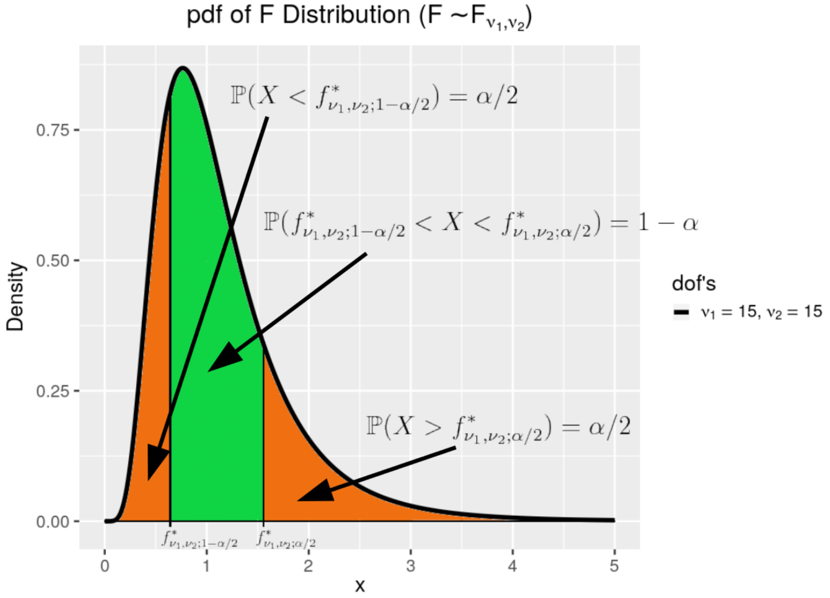
$$\mathbb{P}(F > f^*_{\nu_1,\nu_2;\alpha}) = \alpha$$

NOTE: Do not confuse F-cutoff $f^*_{\nu_1,\nu_2;\alpha}$ with F percentile $f_{\nu_1,\nu_2;\alpha}$:

$$\mathbb{P}(F \leq f_{\nu_1,\nu_2;\alpha}) = \alpha$$

Another name for F-cutoff is F **critical value**.
Finally, notice that $f^*_{\nu_1,\nu_2;\alpha}$ is always **positive**.

F-Cutoffs (Example Plot)



F-Cutoffs (Key Property)

Proposition

Lower-tail F-cutoffs can be determined from appropriate upper-tail F-cutoffs:

$$f^*_{\nu_1,\nu_2;1-\alpha} = \frac{1}{f^*_{\nu_2,\nu_1;\alpha}}$$

IMPORTANT: Notice that the order of the dof's switch!!

PROOF: Beyond scope of course. Take **Mathematical Statistics**.

F-Cutoffs Table $(\alpha = 0.1)$

$(\alpha = \mathbf{0.1})$ **SNEDECOR'S F-CUTOFFS,** $f^*_{\nu_1,\nu_2;\alpha}$ $\mathbb{P}(F > f^*_{\nu_1,\nu_2;\alpha}) = \alpha$, $f^*_{\nu_1,\nu_2;1-\alpha} = \dfrac{1}{f^*_{\nu_2,\nu_1;\alpha}}$

ν_2 \ ν_1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**12**	**15**	**18**	**20**	**30**	**40**	**60**
2	9.000	9.162	9.243	9.293	9.326	9.349	9.367	9.381	9.392	9.408	9.425	9.436	9.441	9.458	9.466	9.475
3	5.462	5.391	5.343	5.309	5.285	5.266	5.252	5.240	5.230	5.216	5.200	5.190	5.184	5.168	5.160	5.151
4	4.325	4.191	4.107	4.051	4.010	3.979	3.955	3.936	3.920	3.896	3.870	3.853	3.844	3.817	3.804	3.790
5	3.780	3.619	3.520	3.453	3.405	3.368	3.339	3.316	3.297	3.268	3.238	3.217	3.207	3.174	3.157	3.140
6	3.463	3.289	3.181	3.108	3.055	3.014	2.983	2.958	2.937	2.905	2.871	2.848	2.836	2.800	2.781	2.762
7	3.257	3.074	2.961	2.883	2.827	2.785	2.752	2.725	2.703	2.668	2.632	2.607	2.595	2.555	2.535	2.514
8	3.113	2.924	2.806	2.726	2.668	2.624	2.589	2.561	2.538	2.502	2.464	2.438	2.425	2.383	2.361	2.339
9	3.006	2.813	2.693	2.611	2.551	2.505	2.469	2.440	2.416	2.379	2.340	2.312	2.298	2.255	2.232	2.208
10	2.924	2.728	2.605	2.522	2.461	2.414	2.377	2.347	2.323	2.284	2.244	2.215	2.201	2.155	2.132	2.107
12	2.807	2.606	2.480	2.394	2.331	2.283	2.245	2.214	2.188	2.147	2.105	2.075	2.060	2.011	1.986	1.960
15	2.695	2.490	2.361	2.273	2.208	2.158	2.119	2.086	2.059	2.017	1.972	1.941	1.924	1.873	1.845	1.817
18	2.624	2.416	2.286	2.196	2.130	2.079	2.038	2.005	1.977	1.933	1.887	1.854	1.837	1.783	1.754	1.723
20	2.589	2.380	2.249	2.158	2.091	2.040	1.999	1.965	1.937	1.892	1.845	1.811	1.794	1.738	1.708	1.677
30	2.489	2.276	2.142	2.049	1.980	1.927	1.884	1.849	1.819	1.773	1.722	1.686	1.667	1.606	1.573	1.538
40	2.440	2.226	2.091	1.997	1.927	1.873	1.829	1.793	1.763	1.715	1.662	1.625	1.605	1.541	1.506	1.467
60	2.393	2.177	2.041	1.946	1.875	1.819	1.775	1.738	1.707	1.657	1.603	1.564	1.543	1.476	1.437	1.395

Use this particular table for:

One-sided $\alpha = 0.1$ F-tests
Two-sided $\alpha = 0.2$ F-tests
80% F-CI's

F-Cutoffs Table $(\alpha = 0.05)$

$(\alpha = \mathbf{0.05})$ **SNEDECOR'S F-CUTOFFS,** $f^*_{\nu_1,\nu_2;\alpha}$ $\mathbb{P}(F > f^*_{\nu_1,\nu_2;\alpha}) = \alpha$, $f^*_{\nu_1,\nu_2;1-\alpha} = \dfrac{1}{f^*_{\nu_2,\nu_1;\alpha}}$

ν_2 \ ν_1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**12**	**15**	**18**	**20**	**30**	**40**	**60**
2	19.00	19.16	19.25	19.30	19.33	19.35	19.37	19.38	19.40	19.41	19.43	19.44	19.45	19.46	19.47	19.48
3	9.552	9.277	9.117	9.013	8.941	8.887	8.845	8.812	8.786	8.745	8.703	8.675	8.660	8.617	8.594	8.572
4	6.944	6.591	6.388	6.256	6.163	6.094	6.041	5.999	5.964	5.912	5.858	5.821	5.803	5.746	5.717	5.688
5	5.786	5.409	5.192	5.050	4.950	4.876	4.818	4.772	4.735	4.678	4.619	4.579	4.558	4.496	4.464	4.431
6	5.143	4.757	4.534	4.387	4.284	4.207	4.147	4.099	4.060	4.000	3.938	3.896	3.874	3.808	3.774	3.740
7	4.737	4.347	4.120	3.972	3.866	3.787	3.726	3.677	3.637	3.575	3.511	3.467	3.445	3.376	3.340	3.304
8	4.459	4.066	3.838	3.687	3.581	3.500	3.438	3.388	3.347	3.284	3.218	3.173	3.150	3.079	3.043	3.005
9	4.256	3.863	3.633	3.482	3.374	3.293	3.230	3.179	3.137	3.073	3.006	2.960	2.936	2.864	2.826	2.787
10	4.103	3.708	3.478	3.326	3.217	3.135	3.072	3.020	2.978	2.913	2.845	2.798	2.774	2.700	2.661	2.621
12	3.885	3.490	3.259	3.106	2.996	2.913	2.849	2.796	2.753	2.687	2.617	2.568	2.544	2.466	2.426	2.384
15	3.682	3.287	3.056	2.901	2.790	2.707	2.641	2.588	2.544	2.475	2.403	2.353	2.328	2.247	2.204	2.160
18	3.555	3.160	2.928	2.773	2.661	2.577	2.510	2.456	2.412	2.342	2.269	2.217	2.191	2.107	2.063	2.017
20	3.493	3.098	2.866	2.711	2.599	2.514	2.447	2.393	2.348	2.278	2.203	2.151	2.124	2.039	1.994	1.946
30	3.316	2.922	2.690	2.534	2.421	2.334	2.266	2.211	2.165	2.092	2.015	1.960	1.932	1.841	1.792	1.740
40	3.232	2.839	2.606	2.449	2.336	2.249	2.180	2.124	2.077	2.003	1.924	1.868	1.839	1.744	1.693	1.637
60	3.150	2.758	2.525	2.368	2.254	2.167	2.097	2.040	1.993	1.917	1.836	1.778	1.748	1.649	1.594	1.534

Use this particular table for:

- One-sided $\alpha = 0.05$ F-tests
- Two-sided $\alpha = 0.1$ F-tests
- 90% F-CI's

F-Cutoffs Table $(\alpha = 0.025)$

$(\alpha = \mathbf{0.025})$ **SNEDECOR'S F-CUTOFFS,** $f^*_{\nu_1,\nu_2;\alpha}$ $\quad \mathbb{P}(F > f^*_{\nu_1,\nu_2;\alpha}) = \alpha, \quad f^*_{\nu_1,\nu_2;1-\alpha} = \dfrac{1}{f^*_{\nu_2,\nu_1;\alpha}}$

ν_2 \ ν_1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**12**	**15**	**18**	**20**	**30**	**40**	**60**
2	39.00	39.17	39.25	39.30	39.33	39.36	39.37	39.39	39.40	39.41	39.43	39.44	39.45	39.46	39.47	39.48
3	16.04	15.44	15.10	14.88	14.73	14.62	14.54	14.47	14.42	14.34	14.25	14.20	14.17	14.08	14.04	13.99
4	10.65	9.979	9.605	9.364	9.197	9.074	8.980	8.905	8.844	8.751	8.657	8.592	8.560	8.461	8.411	8.360
5	8.434	7.764	7.388	7.146	6.978	6.853	6.757	6.681	6.619	6.525	6.428	6.362	6.329	6.227	6.175	6.123
6	7.260	6.599	6.227	5.988	5.820	5.695	5.600	5.523	5.461	5.366	5.269	5.202	5.168	5.065	5.012	4.959
7	6.542	5.890	5.523	5.285	5.119	4.995	4.899	4.823	4.761	4.666	4.568	4.501	4.467	4.362	4.309	4.254
8	6.059	5.416	5.053	4.817	4.652	4.529	4.433	4.357	4.295	4.200	4.101	4.034	3.999	3.894	3.840	3.784
9	5.715	5.078	4.718	4.484	4.320	4.197	4.102	4.026	3.964	3.868	3.769	3.701	3.667	3.560	3.505	3.449
10	5.456	4.826	4.468	4.236	4.072	3.950	3.855	3.779	3.717	3.621	3.522	3.453	3.419	3.311	3.255	3.198
12	5.096	4.474	4.121	3.891	3.728	3.607	3.512	3.436	3.374	3.277	3.177	3.108	3.073	2.963	2.906	2.848
15	4.765	4.153	3.804	3.576	3.415	3.293	3.199	3.123	3.060	2.963	2.862	2.792	2.756	2.644	2.585	2.524
18	4.560	3.954	3.608	3.382	3.221	3.100	3.005	2.929	2.866	2.769	2.667	2.596	2.559	2.445	2.384	2.321
20	4.461	3.859	3.515	3.289	3.128	3.007	2.913	2.837	2.774	2.676	2.573	2.501	2.464	2.349	2.287	2.223
30	4.182	3.589	3.250	3.026	2.867	2.746	2.651	2.575	2.511	2.412	2.307	2.233	2.195	2.074	2.009	1.940
40	4.051	3.463	3.126	2.904	2.744	2.624	2.529	2.452	2.388	2.288	2.182	2.107	2.068	1.943	1.875	1.803
60	3.925	3.343	3.008	2.786	2.627	2.507	2.412	2.334	2.270	2.169	2.061	1.985	1.944	1.815	1.744	1.667

Use this particular table for:

- One-sided $\alpha = 0.025$ F-tests
- Two-sided $\alpha = 0.05$ F-tests
- 95% F-CI's

F-Cutoffs Table $(\alpha = 0.01)$

$(\alpha = \mathbf{0.01})$ **SNEDECOR'S F-CUTOFFS,** $f^*_{\nu_1,\nu_2;\alpha}$ $\quad \mathbb{P}(F > f^*_{\nu_1,\nu_2;\alpha}) = \alpha, \quad f^*_{\nu_1,\nu_2;1-\alpha} = \dfrac{1}{f^*_{\nu_2,\nu_1;\alpha}}$

ν_2 \ ν_1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**12**	**15**	**18**	**20**	**30**	**40**	**60**
2	99.00	99.17	99.25	99.30	99.33	99.36	99.37	99.39	99.40	99.42	99.43	99.44	99.45	99.47	99.47	99.48
3	30.82	29.46	28.71	28.24	27.91	27.67	27.49	27.35	27.23	27.05	26.87	26.75	26.69	26.50	26.41	26.32
4	18.00	16.69	15.98	15.52	15.21	14.98	14.80	14.66	14.55	14.37	14.20	14.08	14.02	13.84	13.75	13.65
5	13.27	12.06	11.39	10.97	10.67	10.46	10.29	10.16	10.05	9.888	9.722	9.610	9.553	9.379	9.291	9.202
6	10.92	9.780	9.148	8.746	8.466	8.260	8.102	7.976	7.874	7.718	7.559	7.451	7.396	7.229	7.143	7.057
7	9.547	8.451	7.847	7.460	7.191	6.993	6.840	6.719	6.620	6.469	6.314	6.209	6.155	5.992	5.908	5.824
8	8.649	7.591	7.006	6.632	6.371	6.178	6.029	5.911	5.814	5.667	5.515	5.412	5.359	5.198	5.116	5.032
9	8.022	6.992	6.422	6.057	5.802	5.613	5.467	5.351	5.257	5.111	4.962	4.860	4.808	4.649	4.567	4.483
10	7.559	6.552	5.994	5.636	5.386	5.200	5.057	4.942	4.849	4.706	4.558	4.457	4.405	4.247	4.165	4.082
12	6.927	5.953	5.412	5.064	4.821	4.640	4.499	4.388	4.296	4.155	4.010	3.909	3.858	3.701	3.619	3.535
15	6.359	5.417	4.893	4.556	4.318	4.142	4.004	3.895	3.805	3.666	3.522	3.423	3.372	3.214	3.132	3.047
18	6.013	5.092	4.579	4.248	4.015	3.841	3.705	3.597	3.508	3.371	3.227	3.128	3.077	2.919	2.835	2.749
20	5.849	4.938	4.431	4.103	3.871	3.699	3.564	3.457	3.368	3.231	3.088	2.989	2.938	2.778	2.695	2.608
30	5.390	4.510	4.018	3.699	3.473	3.304	3.173	3.067	2.979	2.843	2.700	2.600	2.549	2.386	2.299	2.208
40	5.179	4.313	3.828	3.514	3.291	3.124	2.993	2.888	2.801	2.665	2.522	2.421	2.369	2.203	2.114	2.019
60	4.977	4.126	3.649	3.339	3.119	2.953	2.823	2.718	2.632	2.496	2.352	2.251	2.198	2.028	1.936	1.836

Use this particular table for:

- One-sided $\alpha = 0.01$ F-tests
- Two-sided $\alpha = 0.02$ F-tests
- 98% F-CI's

F-Cutoffs Table $(\alpha = 0.005)$

$(\alpha = \mathbf{0.005})$ **SNEDECOR'S F-CUTOFFS,** $f^*_{\nu_1,\nu_2;\alpha}$ $\quad \mathbb{P}(F > f^*_{\nu_1,\nu_2;\alpha}) = \alpha, \quad f^*_{\nu_1,\nu_2;1-\alpha} = \dfrac{1}{f^*_{\nu_2,\nu_1;\alpha}}$

ν_2 \ ν_1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**12**	**15**	**18**	**20**	**30**	**40**	**60**
2	199.0	199.2	199.2	199.3	199.3	199.4	199.4	199.4	199.4	199.4	199.4	199.4	199.4	199.5	199.5	199.5
3	49.80	47.47	46.19	45.39	44.84	44.43	44.13	43.88	43.69	43.39	43.08	42.88	42.78	42.47	42.31	42.15
4	26.28	24.26	23.15	22.46	21.97	21.62	21.35	21.14	20.97	20.70	20.44	20.26	20.17	19.89	19.75	19.61
5	18.31	16.53	15.56	14.94	14.51	14.20	13.96	13.77	13.62	13.38	13.15	12.98	12.90	12.66	12.53	12.40
6	14.54	12.92	12.03	11.46	11.07	10.79	10.57	10.39	10.25	10.03	9.814	9.664	9.589	9.358	9.241	9.122
7	12.40	10.88	10.05	9.522	9.155	8.885	8.678	8.514	8.380	8.176	7.968	7.826	7.754	7.534	7.422	7.309
8	11.04	9.596	8.805	8.302	7.952	7.694	7.496	7.339	7.211	7.015	6.814	6.678	6.608	6.396	6.288	6.177
9	10.11	8.717	7.956	7.471	7.134	6.885	6.693	6.541	6.417	6.227	6.032	5.899	5.832	5.625	5.519	5.410
10	9.427	8.081	7.343	6.872	6.545	6.302	6.116	5.968	5.847	5.661	5.471	5.340	5.274	5.071	4.966	4.859
12	8.510	7.226	6.521	6.071	5.757	5.525	5.345	5.202	5.085	4.906	4.721	4.595	4.530	4.331	4.228	4.123
15	7.701	6.476	5.803	5.372	5.071	4.847	4.674	4.536	4.424	4.250	4.070	3.946	3.883	3.687	3.585	3.480
18	7.215	6.028	5.375	4.956	4.663	4.445	4.276	4.141	4.030	3.860	3.683	3.560	3.498	3.303	3.201	3.096
20	6.986	5.818	5.174	4.762	4.472	4.257	4.090	3.956	3.847	3.678	3.502	3.380	3.318	3.123	3.022	2.916
30	6.355	5.239	4.623	4.228	3.949	3.742	3.580	3.450	3.344	3.179	3.006	2.885	2.823	2.628	2.524	2.415
40	6.066	4.976	4.374	3.986	3.713	3.509	3.350	3.222	3.117	2.953	2.781	2.661	2.598	2.401	2.296	2.184
60	5.795	4.729	4.140	3.760	3.492	3.291	3.134	3.008	2.904	2.742	2.570	2.450	2.387	2.187	2.079	1.962

Use this particular table for:

- One-sided $\alpha = 0.005$ F-tests
- Two-sided $\alpha = 0.01$ F-tests
- 99% F-CI's

PART II:

F-Tests & F-CI's for Comparing Two Normal Population Variances

Comparing Two Population Variances (Motivation)

In most (but not all) situations, a higher variance is not desired:

- Suppose two identical (i.e. same brand & model) DC power supplies convert incoming AC electric current to DC current with the same mean voltage, but their variances in the voltage vastly differ. The power supply bearing the higher variance would be considered less reliable than the other, possibly even defective for an extremely higher variance.
- Employees who do all their online training at work tend to have similar average weekly study durations yet smaller std dev's than those who do most of their online training at home. The smaller variability in the first group of employees allows managers more efficiency in planning and scheduling tasks/projects.
- Two similar investment funds with the same average return rate but differing variances would communicate to the potential investor that the higher-variation fund is higher risk.

This is not surprising at all – we've encountered similar patterns before:

- We prefer point estimators with smaller variances (e.g. UMVUE's.)
- Smaller variances lead to narrower CI's at the same α-level.
- Smaller variances lead to more powerful hypothesis tests with same α.

A Statistic related to the F Distribution

Theorem

Let $\mathbf{X} := (X_1, \ldots, X_m)$ *be a random sample from a* *Normal*(μ_1, σ_1^2) *population.*
Let $\mathbf{Y} := (Y_1, \ldots, Y_n)$ *be a random sample from a* *Normal*(μ_2, σ_2^2) *population.*
Moreover, suppose random samples $\mathbf{X}$ $\&$ $\mathbf{Y}$ *are independent of each other.*

Then the following statistic has an F *distribution:*

$$\frac{S_1^2/\sigma_1^2}{S_2^2/\sigma_2^2} \sim F_{m-1,n-1}$$

PROOF: Beyond scope of course. Take **Mathematical Statistics**.

F-Test for Comparing Normal Pop. Variances (Cutoff Version)

Proposition

Population:	*Two <u>Normal</u> Populations with unknown* σ_+, σ_-
Realized Samples:	$\mathbf{x} := (x_1, x_2, \cdots, x_{n_+})$ *with mean* $\bar{x}$, *std dev* s_+ $\mathbf{y} := (y_1, y_2, \cdots, y_{n_-})$ *with mean* $\bar{y}$, *std dev* s_- *Samples* $\mathbf{x}$ & $\mathbf{y}$ *are independent with* $s_+ \geq s_-$
Test Statistic Values: $W_+(\mathbf{x}, \mathbf{y})$ & $W_-(\mathbf{x}, \mathbf{y})$	$f_+ = \frac{s_+^2}{s_-^2}$, $f_- = \frac{s_-^2}{s_+^2}$, $\nu_+ = n_+ - 1$, $\nu_- = n_- - 1$

HYPOTHESIS TEST:	***REJECTION REGION @ SIGNIF. LVL*** α :
$H_0 : \sigma_+^2 = \sigma_-^2$ *vs.* $H_A : \sigma_+^2 > \sigma_-^2$	$f_+ \geq f^*_{\nu_+, \nu_-; \alpha}$
$H_0 : \sigma_+^2 = \sigma_-^2$ *vs.* $H_A : \sigma_+^2 \neq \sigma_-^2$	$f_- \leq f^*_{\nu_+, \nu_-; 1-\alpha/2}$ *or* $f_+ \geq f^*_{\nu_+, \nu_-; \alpha/2}$

F-Test for Comparing Normal Pop. Variances (P-value Version)

Proposition

Population:	*Two* <u>*Normal*</u> *Populations with unknown* σ_+, σ_-
Realized Samples:	$\mathbf{x} := (x_1, x_2, \cdots, x_{n_+})$ *with mean* $\bar{x}$, *std dev* s_+ $\mathbf{y} := (y_1, y_2, \cdots, y_{n_-})$ *with mean* $\bar{y}$, *std dev* s_- *Samples* $\mathbf{x}$ & $\mathbf{y}$ *are independent with* $s_+ \geq s_-$
Test Statistic Values: $W_+(\mathbf{x}, \mathbf{y})$ & $W_-(\mathbf{x}, \mathbf{y})$	$f_+ = \dfrac{s_+^2}{s_-^2}, \quad f_- = \dfrac{s_-^2}{s_+^2}, \quad \begin{aligned} \nu_+ &= n_+ - 1 \\ \nu_- &= n_- - 1 \end{aligned}$

HYPOTHESIS TEST:	***P-VALUE DETERMINATION:***
$H_0 : \sigma_+^2 = \sigma_-^2$ *vs.* $H_A : \sigma_+^2 > \sigma_-^2$	$1 - \Phi_F(f_+; \nu_+, \nu_-)$
$H_0 : \sigma_+^2 = \sigma_-^2$ *vs.* $H_A : \sigma_+^2 \neq \sigma_-^2$	$\Phi_F(f_-; \nu_-, \nu_+) + [1 - \Phi_F(f_+; \nu_+, \nu_-)]$

F-CI for Normal Pop. Ratio σ_+^2/σ_-^2 (Motivation)

Let $\mathbf{X} := (X_1, \ldots, X_{n_+})$ be a random sample from a Normal(μ_+, σ_+^2) population.
Let $\mathbf{Y} := (Y_1, \ldots, Y_{n_-})$ be a random sample from a Normal(μ_-, σ_-^2) population.
$\mathbf{X}$ & $\mathbf{Y}$ are independent with $s_+^2 \geq s_-^2$, $\nu_+ = n_+ - 1$ & $\nu_- = n_- - 1$.
Then, construct the $100(1-\alpha)\%$ CI for parameter ratio σ_+^2/σ_-^2:

1. Produce a suitable **pivot**: Let $Q(\mathbf{X}, \mathbf{Y}; \sigma_+^2, \sigma_-^2) = (S_+^2/\sigma_+^2)/(S_-^2/\sigma_-^2)$
2. Then the pivot is an F distribution: $Q(\mathbf{X}, \mathbf{Y}; \sigma_+^2, \sigma_-^2) \sim F_{\nu_+,\nu_-}$
3. Find constants $a < b$ such that $\mathbb{P}(a < Q(\mathbf{X}, \mathbf{Y}; \sigma_+^2, \sigma_-^2) < b) = 1 - \alpha$

   Since F_{ν_+,ν_-} pdf is skewed, $\begin{cases} a = f^*_{\nu_+,\nu_-;1-\alpha/2} = 1/f^*_{\nu_-,\nu_+;\alpha/2} \\ b = f^*_{\nu_+,\nu_-;\alpha/2} \end{cases}$
4. Manipulate the inequalities to isolate parameter ratio σ_+^2/σ_-^2:

$$\frac{1}{f^*_{\nu_-,\nu_+;\alpha/2}} < \frac{S_+^2/\sigma_+^2}{S_-^2/\sigma_-^2} < f^*_{\nu_+,\nu_-;\alpha/2} \implies \frac{S_+^2/S_-^2}{f^*_{\nu_+,\nu_-;\alpha/2}} < \frac{\sigma_+^2}{\sigma_-^2} < \frac{f^*_{\nu_-,\nu_+;\alpha/2}}{S_-^2/S_+^2}$$

5. Take independent samples $\mathbf{x} := (x_1, \cdots, x_{n_+})$ & $\mathbf{y} := (y_1, \cdots, y_{n_-})$.
6. Replace point estimators S_+, S_- with s_+, s_- computed from the samples:

$$\frac{s_+^2/s_-^2}{f^*_{\nu_+,\nu_-;\alpha/2}} < \frac{\sigma_+^2}{\sigma_-^2} < \frac{f^*_{\nu_-,\nu_+;\alpha/2}}{s_-^2/s_+^2}$$

F-CI's for σ_+^2/σ_-^2 and σ_+/σ_-

Proposition

Given two normal populations with unknown variances σ_+^2, σ_-^2.
Let $x_1, x_2, \cdots, x_{n_+}$ be a sample taken from 1^{st} population with variance s_+^2.
Let $y_1, y_2, \cdots, y_{n_-}$ be a sample taken from 2^{nd} population with variance s_-^2.
Moreover, the two samples are independent of each other with $s_+^2 \geq s_-^2$.

*Then the $100(1-\alpha)\%$ F**-CI for** σ_+^2/σ_-^2 is*

$$\left(\frac{f_+}{f^*_{\nu_+,\nu_-;\alpha/2}}, \frac{f^*_{\nu_-,\nu_+;\alpha/2}}{f_-}\right) = \left(\frac{s_+^2/s_-^2}{f^*_{\nu_+,\nu_-;\alpha/2}}, \frac{f^*_{\nu_-,\nu_+;\alpha/2}}{s_-^2/s_+^2}\right)$$

*Moreover, the $100(1-\alpha)\%$ F**-CI for** σ_+/σ_- is*

$$\left(\sqrt{\frac{f_+}{f^*_{\nu_+,\nu_-;\alpha/2}}}, \sqrt{\frac{f^*_{\nu_-,\nu_+;\alpha/2}}{f_-}}\right) = \left(\sqrt{\frac{s_+^2/s_-^2}{f^*_{\nu_+,\nu_-;\alpha/2}}}, \sqrt{\frac{f^*_{\nu_-,\nu_+;\alpha/2}}{s_-^2/s_+^2}}\right)$$

where $\nu_+ = n_+ - 1$ and $\nu_- = n_- - 1$

Textbook Logistics for Section 9.5

CONCEPT	TEXTBOOK NOTATION	SLIDES/OUTLINE NOTATION
Probability of Event $A \subseteq \Omega$	$P(A)$	$\mathbb{P}(A)$
Expected Value of rv X	$E(X)$	$\mathbb{E}[X]$
Variance of rv X	$V(X)$	$\mathbb{V}[X]$
Alternative Hypothesis	H_a	H_A
F-Cutoffs	F_{α,ν_1,ν_2}	$f^*_{\nu_1,\nu_2;\alpha}$
F-Test Stat Value(s)	f	$f_-,\ f_+$

Fin.