Motivation: Chi-Square Distributions

Engineering Statistics

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TTU

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PART I:

Motivation

χ^2_1 from $N(0,1)$ cdf

Degrees of Freedom

χ^2 Distribution (Motivation)

Motivation:

1. Typical errors are distributed as Normal(μ, σ^2).
2. Standardized errors are distributed as $N(0, 1)$.
3. If $Z_1, \cdots, Z_n \overset{IID}{\sim} N(0, 1)$, then their sum of squares $\sum_k Z_k^2$ is of interest...
   ...but how is this sum of squares of standard normal rv's distributed?

χ_1^2 Distribution from Standard Normal cdf

Theorem

(χ_1^2 Theorem – CHISQ1THM)

$Z \sim N(0,1) \implies Y := Z^2 \sim \chi_1^2$ *where* $f_Y(y) = \frac{1}{\sqrt{2\pi}} y^{-1/2} e^{-y/2} = \frac{y^{1/2-1} e^{-y/2}}{2^{1/2} \cdot \Gamma(1/2)} \leftarrow \Gamma_{1/2,2}$ *pdf*

*This distribution of Z^2 is called the **chi-square distribution with one degree of freedom**.*

<u>PROOF:</u>

$$\begin{matrix} Z \sim N(0,1) \\ \Omega_Z = (-\infty, \infty) \end{matrix} \implies f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \implies \Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-t^2/2}\, dt$$

Apply change of random variables (CRV): Let $Y := Z^2$

$$F_Y(y) = \mathbb{P}[Y \le y] \overset{CRV}{=} \mathbb{P}\left[Z^2 \le y\right] = \mathbb{P}\left[-\sqrt{y} \le Z \le \sqrt{y}\right] = \mathbb{P}\left[Z \le \sqrt{y}\right] - \mathbb{P}\left[Z \le -\sqrt{y}\right]$$

$$\overset{N(0,1)}{=} \Phi(\sqrt{y}) - \Phi(-\sqrt{y}) = \int_{-\infty}^{\sqrt{y}} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\, dt - \int_{-\infty}^{-\sqrt{y}} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\, dt = \int_{-\sqrt{y}}^{\sqrt{y}} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\, dt$$

$$\overset{SYMI}{=} 2 \cdot \int_0^{\sqrt{y}} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\, dt \overset{CV}{=} 2 \cdot \int_0^{y} \frac{1}{\sqrt{2\pi}} e^{-x/2} \cdot \frac{1}{2\sqrt{x}}\, dx = \int_0^{y} \frac{1}{\sqrt{2\pi}} x^{-1/2} e^{-x/2}\, dx$$

(CV): Let $x := t^2 \implies dx = 2t\, dt \implies dt = \frac{1}{2\sqrt{x}}\, dx \implies \begin{cases} x(\sqrt{y}) = (\sqrt{y})^2 = y \\ x(0) = (0)^2 = 0 \end{cases}$

χ_1^2 Distribution from Standard Normal cdf

Theorem

(χ_1^2 Theorem – CHISQ1THM)

$Z \sim N(0,1) \implies Y := Z^2 \sim \chi_1^2$ *where* $f_Y(y) = \dfrac{1}{\sqrt{2\pi}} y^{-1/2} e^{-y/2} = \dfrac{y^{1/2-1} e^{-y/2}}{2^{1/2} \cdot \Gamma(1/2)} \leftarrow \Gamma_{1/2,2}$ *pdf*

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$$\begin{matrix} Z \sim N(0,1) \\ \Omega_Z = (-\infty,\infty) \end{matrix} \implies f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \implies \Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-t^2/2}\, dt$$

Change of random variables (CRV): Let $Y := Z^2$

$$\implies \text{cdf of } Y \text{ is } F_Y(y) = \int_0^y \frac{1}{\sqrt{2\pi}} x^{-1/2} e^{-x/2}\, dx = \int_0^y \frac{1}{2^{1/2} \cdot \Gamma(1/2)} x^{1/2-1} e^{-x/2}\, dx$$

Limits of integration and integrand for cdf of $Y \implies \Omega_Y = (0,\infty)$

$$\implies \text{pdf of } Y \text{ is } f_Y(y) = \frac{1}{2^{1/2} \cdot \Gamma(1/2)} y^{1/2-1} e^{-y/2} \text{ which is the pdf of Gamma}(\alpha = 1/2, \beta = 2)$$

$$\therefore \begin{matrix} Z \sim N(0,1) \\ \Omega_Z = (-\infty,\infty) \end{matrix} \implies \begin{matrix} Y := Z^2 \sim \chi_1^2 \\ \Omega_Y = (0,\infty) \end{matrix} \quad \square$$

R.V. Hogg, A.T. Craig, *Introduction to Mathematical Statistics*, 5^{th} Ed, Prentice Hall, 1995. (§6.6)
R.J. Larsen, M.L. Marx, *An Intro to Mathematical Statistics...*, 2^{nd} Ed, Prentice Hall, 1986. (§7.5)
A. Hald, *Statistical Theory with Engineering Applications*, Wiley, 1952. (§10.4)

Degrees of Freedom (General Motivation)

“Suppose you are asked to write 3 numbers with no restrictions upon them. You have complete freedom of choice in regard to all 3. There are 3 degrees of freedom.”†

“Now suppose you are asked to write 3 numbers with the restriction that their sum is to be some particular value, say 20. You cannot now choose all 3 freely, but as soon as 2 have been chosen the third is determined. Your choices are governed by the necessary relation $X_1 + X_2 + X_3 = 20$. In this situation there are only 2 degrees of freedom. The number of variables is 3, but the number of restrictions upon them is 1, and the number of ’free’ variables, or independent choices, is $3 - 1 = 2$.”†

†H.M. Walker, J. Lev, *Statistical Inference*, Henry Holt and Company, 1953. (Ch 4)

Degrees of Freedom (General Motivation)

"Now suppose you are asked to write 5 numbers such that their sum is 30 and also such that the sum of the first two is 18. There are 5 variables but you do not have freedom of choice with respect to all 5. You cannot write 5 numbers arbitrarily and have them conform to the 2 restrictions that:

$$X_1 + X_2 = 18 \quad \text{and} \quad X_1 + X_2 + X_3 + X_4 + X_5 = 30$$

As soon as you select X_1, then $X_2 = 18 - X_1$ and is completely determined. Since $X_3 + X_4 + X_5 = 30 - 18 = 12$, only two of the numbers, X_3, X_4, and X_5, can be freely chosen. As one of the numbers X_1 and X_2 can be freely chosen there are 3 free choices. The number of degrees of freedom is $n = 5 - 2 = 3$."[†]

"In every statistical problem in which degrees of freedom are involved it is necessary to determine the number of **free variables** by first noting the total number of variables and reducing that number by the number of **independent restrictions** upon them. In the preceding paragraph, for instance, one might think there are 3 restrictions, namely:

$$X_1 + X_2 = 18 \quad X_3 + X_4 + X_5 = 12 \quad X_1 + X_2 + X_3 + X_4 + X_5 = 30$$

However only two of these are independent, since any one of them can be deduced from the other two."[†]

[†]H.M. Walker, J. Lev, *Statistical Inference*, Henry Holt and Company, 1953. (Ch 4)

χ^2 Distribution (Degrees of Freedom)

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1. Typical errors are distributed as Normal(μ, σ^2).
2. Standardized errors are distributed as $N(0, 1)$.
3. If $Z_1, \cdots, Z_n \overset{IID}{\sim} N(0, 1)$, then their sum of squares $\sum_k Z_k^2$ is of interest...
   ...but how is this sum of squares of standard normal rv's distributed?

Since $Z_1, \cdots, Z_n$ are all independent,
there are no constraints imposed,
so they are n degrees of freedom.

$\sum_k Z_k^2 \sim \chi_n^2$
(chi-square with n degrees of freedom)

PART II:

Shearing Sum Mapping

χ_2^2 from χ_1^2 & χ_1^2 pdf's
χ_3^2 from χ_2^2 & χ_1^2 pdf's
χ_4^2 from χ_3^2 & χ_1^2 pdf's
χ_5^2 from χ_4^2 & χ_1^2 pdf's
χ_6^2 from χ_5^2 & χ_1^2 pdf's
χ_7^2 from χ_6^2 & χ_1^2 pdf's

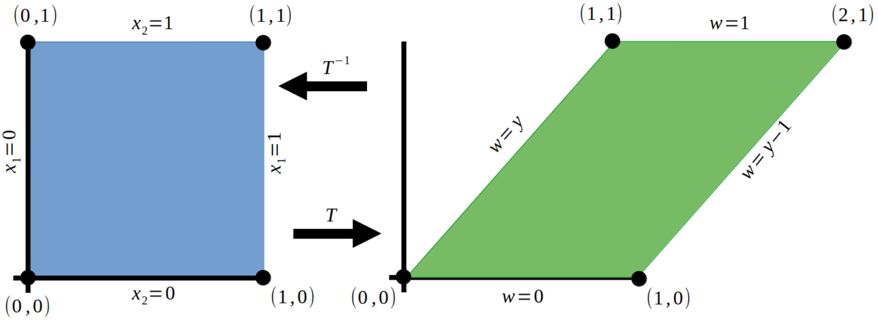
χ_{2k}^2 pattern
χ_{2k+1}^2 pattern

χ_{2k}^2	from	χ_{2k-1}^2	&	χ_1^2	pdf's
χ_{2k+1}^2	from	χ_{2k}^2	&	χ_1^2	pdf's

Adding Two RV's: Shearing Sum Mapping

$$T : \begin{cases} Y := X_1 + X_2 \\ W := X_2 \end{cases} \iff T^{-1} : \begin{cases} X_1 := Y - W \\ X_2 := W \end{cases}$$

(x_1, x_2)	$(y, w) = T(x_1, x_2)$	$(x_1, x_2) = T^{-1}(y, w)$
$(0,0)$	$T(0,0) = (0+0,0) = (0,0)$	$T^{-1}(0,0) = (0-0,0) = (0,0)$
$(1,0)$	$T(1,0) = (1+0,0) = (1,0)$	$T^{-1}(1,0) = (1-0,0) = (1,0)$
$(0,1)$	$T(0,1) = (0+1,1) = (1,1)$	$T^{-1}(1,1) = (1-1,1) = (0,1)$
$(1,1)$	$T(1,1) = (1+1,1) = (2,1)$	$T^{-1}(2,1) = (2-1,1) = (1,1)$

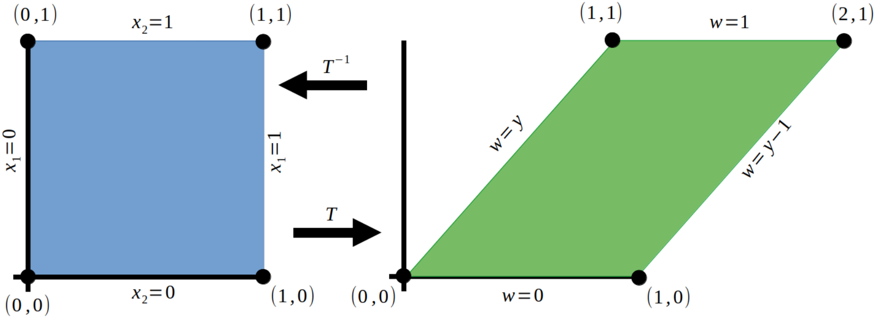


Adding Two RV's: Shearing Sum Mapping

$$T : \begin{cases} Y & := & X_1 + X_2 \\ W & := & X_2 \end{cases} \iff T^{-1} : \begin{cases} X_1 & := & Y - W \\ X_2 & := & W \end{cases}$$

Jacobian of inverse mapping $$J_{T^{-1}} \equiv J = \begin{bmatrix} \frac{\partial X_1}{\partial Y} & \frac{\partial X_1}{\partial W} \\ \frac{\partial X_2}{\partial Y} & \frac{\partial X_2}{\partial W} \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \implies |J| = 1$$

Jacobian equals one meaning the image of this shearing sum mapping of the unit square (with unit area) is the parallelogram <u>also with the same unit area</u>.



Adding Two RV's: Shearing Sum Mapping

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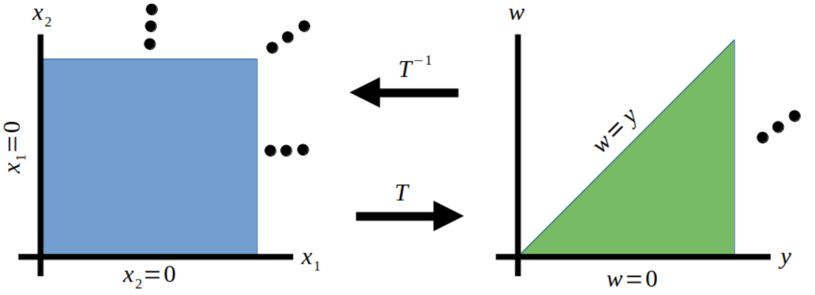

Joint Support Ω_{X_1,X_2}

Joint Support $\Omega_{Y,W}$

$\Omega_{Y,W}$ is V-simple $\implies \Omega_1^V = \Omega_{Y,W}$

Top BC $\overline{\partial}\Omega_1^V = \{(y,w) : w = y\}$

Btm BC $\underline{\partial}\Omega_1^V = \{(y,w) : w = 0\}$

χ_2^2 from χ_1^2 & χ_1^2 pdf's

Theorem

(χ_2^2 Theorem – CHISQ2THM)

$X_1, X_2 \overset{IID}{\sim} \chi_1^2 \implies Y := X_1 + X_2 \sim \chi_2^2$ *where* $f_Y(y) = \frac{1}{2}e^{-y/2} = \frac{y^{2/2-1}e^{-y/2}}{2^{2/2}\cdot\Gamma(2/2)} \leftarrow \Gamma_{1,2}$ *pdf*

This distribution of Y is called the ***chi-square distribution with two degrees of freedom.***

<u>PROOF:</u>

pdf's $f_{X_1}(x_1) = \frac{x_1^{-1/2}e^{-x_1/2}}{\sqrt{2\pi}}, \; f_{X_2}(x_2) = \frac{x_2^{-1/2}e^{-x_2/2}}{\sqrt{2\pi}} \implies$ Supports $\Omega_{X_1} = \Omega_{X_2} = (0,\infty)$

Joint pdf $f_{X_1,X_2}(x_1,x_2) \overset{IND}{=} f_{X_1}(x_1)\cdot f_{X_2}(x_2) = \frac{x_1^{-1/2}e^{-x_1/2}}{\sqrt{2\pi}} \cdot \frac{x_2^{-1/2}e^{-x_2/2}}{\sqrt{2\pi}} = \frac{(x_1x_2)^{-1/2}e^{-(x_1+x_2)/2}}{2\pi}$

Joint Support $\Omega_{X_1,X_2} \overset{IND}{=} \Omega_{X_1} \times \Omega_{X_2} = \{(x_1,x_2) : x_1 \in \Omega_{X_1}, x_2 \in \Omega_{X_2}\} = \{(x_1,x_2) : x_1 > 0, x_2 > 0\}$

Change of random variables (CRV):

$$\begin{matrix} Y := X_1 + X_2 \\ W := X_2 \end{matrix} \iff \begin{matrix} X_1 = Y - W \\ X_2 = W \end{matrix} \implies J = \begin{bmatrix} \frac{\partial X_1}{\partial Y} & \frac{\partial X_1}{\partial W} \\ \frac{\partial X_2}{\partial Y} & \frac{\partial X_2}{\partial W} \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \implies |J| = 1$$

$\implies$ Joint pdf $f_{Y,W}(y,w) \overset{CRV}{=} f_{X_1,X_2}(y-w,w)\cdot|J| = \frac{(y-w)^{-1/2}w^{-1/2}e^{-y/2}}{2\pi}$

$\implies$ Joint Support $\Omega_{Y,W} = \{(y,w) : 0 < y < \infty, \; 0 < w < y\}$

χ^2_2 from χ^2_1 & χ^2_1 pdf's

Theorem

(χ^2_2 Theorem – CHISQ2THM)

$X_1, X_2 \overset{IID}{\sim} \chi^2_1 \implies Y := X_1 + X_2 \sim \chi^2_2$ *where* $f_Y(y) = \frac{1}{2} e^{-y/2} = \frac{y^{2/2-1} e^{-y/2}}{2^{2/2} \cdot \Gamma(2/2)} \leftarrow \Gamma_{1,2}$ *pdf*

*This distribution of Y is called the **chi-square distribution with two degrees of freedom**.*

<u>PROOF:</u> $\implies$ Joint pdf $f_{Y,W}(y,w) \overset{CRV}{=} f_{X_1,X_2}(y-w, w) \cdot |J| = \frac{(y-w)^{-1/2} w^{-1/2} e^{-y/2}}{2\pi}$

$\implies$ Joint Support $\Omega_{Y,W} = \{(y,w) : 0 < y < \infty,\ 0 < w < y\}$

$\implies$ Marginal pdf $f_Y(y) = \int_{\underline{\partial}\Omega_1^V}^{\overline{\partial}\Omega_1^V} f_{Y,W}(y,w)\, dw = \int_0^y \frac{(y-w)^{-1/2} w^{-1/2} e^{-y/2}}{2\pi}\, dw$

$$f_Y(y) = \frac{e^{-y/2}}{2\pi} \int_0^y \frac{dw}{w^{1/2}(y-w)^{1/2}} \quad (CV)\ w := y \sin^2\theta \iff \begin{cases} y - w & = & y\cos^2\theta \\ dw & = & 2y\sin\theta\cos\theta\, d\theta \\ w = y & \iff & \theta = \pi/2 \\ w = 0 & \iff & \theta = 0 \end{cases}$$

$$f_Y(y) \overset{CV}{=} \frac{e^{-y/2}}{2\pi} \cdot \int_0^{\pi/2} \frac{2y\sin\theta\cos\theta\, d\theta}{y^{1/2}\sin\theta \cdot y^{1/2}\cos\theta} = \frac{e^{-y/2}}{\pi} \cdot \int_0^{\pi/2} d\theta = \frac{e^{-y/2}}{\pi} \cdot [\theta]_{\theta=0}^{\theta=\pi/2}$$

$$f_Y(y) \overset{FTC}{=} \frac{e^{-y/2}}{\pi} \cdot \left[\frac{\pi}{2} - 0\right] = \frac{1}{2} e^{-y/2} \qquad \therefore f_Y(y) = \frac{1}{2} e^{-y/2} = \frac{y^{2/2-1} e^{-y/2}}{2^{2/2} \cdot \Gamma(2/2)} \qquad \square$$

H.W. Alexander, *Elements of Mathematical Statistics*, Wiley, 1961. (Ch5, §37)

χ_3^2 from χ_2^2 & χ_1^2 pdf's

Theorem

(χ_3^2 Theorem – CHISQ3THM)

$X_1, X_2 \overset{IND}{\sim} \chi_1^2, \chi_2^2 \implies Y := X_1 + X_2 \sim \chi_3^2$ *where* $f_Y(y) = \dfrac{y^{1/2} e^{-y/2}}{\sqrt{2\pi}} = \dfrac{y^{3/2-1} e^{-y/2}}{2^{3/2} \cdot \Gamma(3/2)} \leftarrow \Gamma_{3/2,2}$ *pdf*

*This distribution of Y is called the **chi-square distribution with three degrees of freedom**.*

PROOF:

pdf's $f_{X_1}(x_1) = \frac{x_1^{-1/2} e^{-x_1/2}}{\sqrt{2\pi}}, \ f_{X_2}(x_2) = \frac{e^{-x_2/2}}{2} \implies$ Supports $\Omega_{X_1} = \Omega_{X_2} = (0, \infty)$

Joint pdf $f_{X_1,X_2}(x_1, x_2) \overset{IND}{=} f_{X_1}(x_1) \cdot f_{X_2}(x_2) = \dfrac{x_1^{-1/2} e^{-x_1/2}}{\sqrt{2\pi}} \cdot \dfrac{e^{-x_2/2}}{2} = \dfrac{x_1^{-1/2} e^{-(x_1+x_2)/2}}{2\sqrt{2\pi}}$

Joint Support $\Omega_{X_1,X_2} \overset{IND}{=} \Omega_{X_1} \times \Omega_{X_2} = \{(x_1, x_2) : x_1 \in \Omega_{X_1}, x_2 \in \Omega_{X_2}\} = \{(x_1, x_2) : x_1 > 0, x_2 > 0\}$

Change of random variables (CRV):

$$\begin{array}{l} Y := X_1 + X_2 \\ W := X_2 \end{array} \iff \begin{array}{l} X_1 = Y - W \\ X_2 = W \end{array} \implies J = \begin{bmatrix} \frac{\partial X_1}{\partial Y} & \frac{\partial X_1}{\partial W} \\ \frac{\partial X_2}{\partial Y} & \frac{\partial X_2}{\partial W} \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \implies |J| = 1$$

$\implies$ Joint pdf $f_{Y,W}(y, w) \overset{CRV}{=} f_{X_1,X_2}(y - w, w) \cdot |J| = \dfrac{(y-w)^{-1/2} e^{-y/2}}{2\sqrt{2\pi}}$

$\implies$ Joint Support $\Omega_{Y,W} = \{(y, w) : 0 < y < \infty, \ 0 < w < y\}$

χ_3^2 from χ_2^2 & χ_1^2 pdf's

Theorem

(χ_3^2 Theorem – CHISQ3THM)

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PROOF:

$\implies$ Joint pdf $f_{Y,W}(y,w) \overset{CRV}{=} f_{X_1,X_2}(y-w, w) \cdot |J| = \frac{(y-w)^{-1/2} e^{-y/2}}{2\sqrt{2\pi}}$

$\implies$ Joint Support $\Omega_{Y,W} = \{(y,w) : 0 < y < \infty,\ 0 < w < y\}$

$\implies$ Marginal pdf $f_Y(y) = \int_{\underline{\partial}\Omega_1^Y}^{\overline{\partial}\Omega_1^Y} f_{Y,W}(y,w)\, dw = \int_0^y \frac{(y-w)^{-1/2} e^{-y/2}}{2\sqrt{2\pi}}\, dw$

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$$f_Y(y) \overset{CV}{=} \frac{e^{-y/2}}{2\sqrt{2\pi}} \cdot \int_0^{\pi/2} \frac{2y\sin\theta\cos\theta\, d\theta}{y^{1/2}\cos\theta} = \frac{y^{1/2} e^{-y/2}}{\sqrt{2\pi}} \cdot \int_0^{\pi/2} \sin\theta\, d\theta = \frac{y^{1/2} e^{-y/2}}{\sqrt{2\pi}} \cdot \Big[-\cos\theta\Big]_{\theta=0}^{\theta=\pi/2}$$

$$f_Y(y) \overset{FTC}{=} \frac{y^{1/2} e^{-y/2}}{\sqrt{2\pi}} \cdot [0 - (-1)] = \frac{1}{\sqrt{2\pi}} y^{1/2} e^{-y/2} \quad \therefore f_Y(y) = \frac{1}{\sqrt{2\pi}} y^{1/2} e^{-y/2} = \frac{y^{3/2-1} e^{-y/2}}{2^{3/2} \cdot \Gamma(3/2)} \quad \square$$

χ_4^2 from χ_3^2 & χ_1^2 pdf's

Theorem

(χ_4^2 Theorem – CHISQ4THM)

$X_1, X_2 \overset{IND}{\sim} \chi_1^2, \chi_3^2 \implies Y := X_1 + X_2 \sim \chi_4^2$ *where* $f_Y(y) = \frac{1}{4} y e^{-y/2} = \frac{y^{4/2-1} e^{-y/2}}{2^{4/2} \cdot \Gamma(4/2)} \leftarrow \Gamma_{2,2}$ *pdf*

*This distribution of Y is called the **chi-square distribution with four degrees of freedom**.*

PROOF:

pdf's $f_{X_1}(x_1) = \frac{x_1^{-1/2} e^{-x_1/2}}{\sqrt{2\pi}}, \; f_{X_2}(x_2) = \frac{x_2^{1/2} e^{-x_2/2}}{\sqrt{2\pi}} \implies$ Supports $\Omega_{X_1} = \Omega_{X_2} = (0, \infty)$

Joint pdf $f_{X_1,X_2}(x_1, x_2) \overset{IND}{=} f_{X_1}(x_1) \cdot f_{X_2}(x_2) = \frac{x_1^{-1/2} e^{-x_1/2}}{\sqrt{2\pi}} \cdot \frac{x_2^{1/2} e^{-x_2/2}}{\sqrt{2\pi}} = \frac{x_1^{-1/2} x_2^{1/2} e^{-(x_1+x_2)/2}}{2\pi}$

Joint Support $\Omega_{X_1,X_2} \overset{IND}{=} \Omega_{X_1} \times \Omega_{X_2} = \{(x_1, x_2) : x_1 \in \Omega_{X_1}, x_2 \in \Omega_{X_2}\} = \{(x_1, x_2) : x_1 > 0, x_2 > 0\}$

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$\implies$ Joint pdf $f_{Y,W}(y, w) \overset{CRV}{=} f_{X_1,X_2}(y - w, w) \cdot |J| = \frac{(y-w)^{-1/2} w^{1/2} e^{-y/2}}{2\pi}$

$\implies$ Joint Support $\Omega_{Y,W} = \{(y, w) : 0 < y < \infty, \; 0 < w < y\}$

χ_4^2 from χ_3^2 & χ_1^2 pdf's

Theorem

(χ_4^2 Theorem – CHISQ4THM)

$X_1, X_2 \overset{IND}{\sim} \chi_1^2, \chi_3^2 \implies Y := X_1 + X_2 \sim \chi_4^2$ *where* $f_Y(y) = \frac{1}{4} y e^{-y/2} = \frac{y^{4/2-1} e^{-y/2}}{2^{4/2} \cdot \Gamma(4/2)} \leftarrow \Gamma_{2,2}$ *pdf*

This distribution of Y is called the ***chi-square distribution with four degrees of freedom****.*

PROOF:

$\implies$ Joint pdf $f_{Y,W}(y,w) \overset{CRV}{=} f_{X_1,X_2}(y-w, w) \cdot |J| = \dfrac{(y-w)^{-1/2} w^{1/2} e^{-y/2}}{2\pi}$

$\implies$ Joint Support $\Omega_{Y,W} = \{(y,w) : 0 < y < \infty,\ 0 < w < y\}$

$\implies$ Marginal pdf $f_Y(y) = \displaystyle\int_{\underline{\partial}\Omega_1^V}^{\overline{\partial}\Omega_1^V} f_{Y,W}(y,w)\, dw = \int_0^y \frac{(y-w)^{-1/2} w^{1/2} e^{-y/2}}{2\pi}\, dw$

$$f_Y(y) = \frac{e^{-y/2}}{2\pi} \int_0^y \frac{w^{1/2}\, dw}{(y-w)^{1/2}} \quad (CV)\ w := y \sin^2\theta \iff \begin{cases} y - w & = & y\cos^2\theta \\ dw & = & 2y \sin\theta\cos\theta\, d\theta \\ w = y & \iff & \theta = \pi/2 \\ w = 0 & \iff & \theta = 0 \end{cases}$$

$$f_Y(y) \overset{CV}{=} \frac{e^{-y/2}}{2\pi} \cdot \int_0^{\pi/2} \frac{(y^{1/2}\sin\theta)(2y\sin\theta\cos\theta\, d\theta)}{y^{1/2}\cos\theta}$$

$$f_Y(y) = \frac{y e^{-y/2}}{\pi} \cdot \int_0^{\pi/2} \sin^2\theta\, d\theta$$

Theorem

(χ_4^2 Theorem – CHISQ4THM)

$X_1, X_2 \overset{IND}{\sim} \chi_1^2, \chi_3^2 \implies Y := X_1 + X_2 \sim \chi_4^2$ *where* $f_Y(y) = \frac{1}{4} y e^{-y/2} = \frac{y^{4/2-1} e^{-y/2}}{2^{4/2} \cdot \Gamma(4/2)} \leftarrow \Gamma_{2,2}$ *pdf*

*This distribution of Y is called the **chi-square distribution with four degrees of freedom**.*

PROOF:

$\implies$ Joint pdf $f_{Y,W}(y,w) \overset{CRV}{=} f_{X_1,X_2}(y-w, w) \cdot |J| = \frac{(y-w)^{-1/2} w^{1/2} e^{-y/2}}{2\pi}$

$\implies$ Joint Support $\Omega_{Y,W} = \{(y,w) : 0 < y < \infty,\ 0 < w < y\}$

$\implies$ Marginal pdf $f_Y(y) = \int_{\underline{\partial}\Omega_1^V}^{\overline{\partial}\Omega_1^V} f_{Y,W}(y,w)\, dw = \int_0^y \frac{(y-w)^{-1/2} w^{1/2} e^{-y/2}}{2\pi}\, dw$

$$f_Y(y) = \frac{y e^{-y/2}}{\pi} \cdot \int_0^{\pi/2} \sin^2\theta\, d\theta \overset{TRIG}{=} \frac{y e^{-y/2}}{\pi} \cdot \int_0^{\pi/2} \frac{1}{2} - \frac{\cos 2\theta}{2}\, d\theta = \frac{y e^{-y/2}}{\pi} \cdot \left[\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]_{\theta=0}^{\theta=\pi/2}$$

$$f_Y(y) \overset{FTC}{=} \frac{y e^{-y/2}}{\pi} \cdot \left[\left(\frac{\pi}{4} - 0\right) - \left(\frac{0}{2} - 0\right)\right] = \frac{1}{4} y e^{-y/2} \quad \therefore f_Y(y) = \frac{1}{4} y e^{-y/2} = \frac{y^{4/2-1} e^{-y/2}}{2^{4/2} \cdot \Gamma(4/2)} \qquad \square$$

χ^2_5 from χ^2_4 & χ^2_1 pdf's

Theorem

(χ^2_5 Theorem – CHISQ5THM)

$X_1, X_2 \overset{IND}{\sim} \chi^2_1, \chi^2_4 \implies Y := X_1 + X_2 \sim \chi^2_5$ *where* $f_Y(y) = \dfrac{y^{3/2}e^{-y/2}}{3\sqrt{2\pi}} = \dfrac{y^{5/2-1}e^{-y/2}}{2^{5/2}\cdot\Gamma(5/2)} \leftarrow \Gamma_{5/2,2}$ *pdf*

This distribution of Y is called the ***chi-square distribution with five degrees of freedom.***

PROOF:

pdf's $f_{X_1}(x_1) = \dfrac{x_1^{-1/2}e^{-x_1/2}}{\sqrt{2\pi}}$, $f_{X_2}(x_2) = \dfrac{1}{4}x_2e^{-x_2/2} \implies$ Supports $\Omega_{X_1} = \Omega_{X_2} = (0,\infty)$

Joint pdf $f_{X_1,X_2}(x_1,x_2) \overset{IND}{=} f_{X_1}(x_1)\cdot f_{X_2}(x_2) = \dfrac{x_1^{-1/2}e^{-x_1/2}}{\sqrt{2\pi}}\cdot\dfrac{1}{4}x_2e^{-x_2/2} = \dfrac{x_1^{-1/2}x_2e^{-(x_1+x_2)/2}}{4\sqrt{2\pi}}$

Joint Support $\Omega_{X_1,X_2} \overset{IND}{=} \Omega_{X_1}\times\Omega_{X_2} = \{(x_1,x_2) : x_1\in\Omega_{X_1}, x_2\in\Omega_{X_2}\} = \{(x_1,x_2) : x_1 > 0, x_2 > 0\}$

Change of random variables (CRV):

$$\begin{array}{l} Y := X_1 + X_2 \\ W := X_2 \end{array} \iff \begin{array}{l} X_1 = Y - W \\ X_2 = W \end{array} \implies J = \begin{bmatrix} \frac{\partial X_1}{\partial Y} & \frac{\partial X_1}{\partial W} \\ \frac{\partial X_2}{\partial Y} & \frac{\partial X_2}{\partial W} \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \implies |J| = 1$$

$\implies$ Joint pdf $f_{Y,W}(y,w) \overset{CRV}{=} f_{X_1,X_2}(y-w,w)\cdot|J| = \dfrac{(y-w)^{-1/2}we^{-y/2}}{4\sqrt{2\pi}}$

$\implies$ Joint Support $\Omega_{Y,W} = \{(y,w) : 0 < y < \infty,\ 0 < w < y\}$

χ^2_5 from χ^2_4 & χ^2_1 pdf's

Theorem

(χ^2_5 Theorem – CHISQ5THM)

$X_1, X_2 \overset{IND}{\sim} \chi^2_1, \chi^2_4 \implies Y := X_1 + X_2 \sim \chi^2_5$ *where* $f_Y(y) = \dfrac{y^{3/2}e^{-y/2}}{3\sqrt{2\pi}} = \dfrac{y^{5/2-1}e^{-y/2}}{2^{5/2}\cdot\Gamma(5/2)} \leftarrow \Gamma_{5/2,2}$ *pdf*

*This distribution of Y is called the **chi-square distribution with five degrees of freedom**.*

PROOF:

$\implies$ Joint pdf $f_{Y,W}(y,w) \overset{CRV}{=} f_{X_1,X_2}(y-w, w)\cdot|J| = \dfrac{(y-w)^{-1/2}we^{-y/2}}{4\sqrt{2\pi}}$

$\implies$ Joint Support $\Omega_{Y,W} = \{(y,w) : 0 < y < \infty,\ 0 < w < y\}$

$\implies$ Marginal pdf $f_Y(y) = \displaystyle\int_{\underline{\partial}\Omega_1^V}^{\overline{\partial}\Omega_1^V} f_{Y,W}(y,w)\,dw = \int_0^y \frac{(y-w)^{-1/2}we^{-y/2}}{4\sqrt{2\pi}}\,dw$

$$f_Y(y) = \frac{e^{-y/2}}{4\sqrt{2\pi}}\int_0^y \frac{w\,dw}{(y-w)^{1/2}} \quad (CV)\ w := y\sin^2\theta \iff \begin{cases} y-w & = & y\cos^2\theta \\ dw & = & 2y\sin\theta\cos\theta\,d\theta \\ w=y & \iff & \theta = \pi/2 \\ w=0 & \iff & \theta = 0 \end{cases}$$

$$f_Y(y) \overset{CV}{=} \frac{e^{-y/2}}{4\sqrt{2\pi}}\cdot\int_0^{\pi/2}\frac{(y\sin^2\theta)(2y\sin\theta\cos\theta\,d\theta)}{y^{1/2}\cos\theta}$$

$$f_Y(y) = \frac{y^{3/2}e^{-y/2}}{2\sqrt{2\pi}}\cdot\int_0^{\pi/2}\sin^3\theta\,d\theta$$

χ^2_5 from χ^2_4 & χ^2_1 pdf's

Theorem

(χ^2_5 Theorem – CHISQ5THM)

$X_1, X_2 \overset{IND}{\sim} \chi^2_1, \chi^2_4 \implies Y := X_1 + X_2 \sim \chi^2_5$ *where* $f_Y(y) = \dfrac{y^{3/2}e^{-y/2}}{3\sqrt{2\pi}} = \dfrac{y^{5/2-1}e^{-y/2}}{2^{5/2}\cdot\Gamma(5/2)} \leftarrow \Gamma_{5/2,2}$ *pdf*

This distribution of Y is called the ***chi-square distribution with five degrees of freedom.***

PROOF:

$\implies$ Joint pdf $f_{Y,W}(y,w) \overset{CRV}{=} f_{X_1,X_2}(y-w,w)\cdot|J| = \dfrac{(y-w)^{-1/2}we^{-y/2}}{4\sqrt{2\pi}}$

$\implies$ Joint Support $\Omega_{Y,W} = \{(y,w) : 0 < y < \infty,\ 0 < w < y\}$

$\implies$ Marginal pdf $f_Y(y) = \displaystyle\int_{\underline{\partial}\Omega_1^V}^{\overline{\partial}\Omega_1^V} f_{Y,W}(y,w)\,dw = \int_0^y \frac{(y-w)^{-1/2}we^{-y/2}}{4\sqrt{2\pi}}\,dw$

$$f_Y(y) = \frac{y^{3/2}e^{-y/2}}{2\sqrt{2\pi}}\cdot\int_0^{\pi/2}\sin^3\theta\,d\theta \overset{TRIG}{=} \frac{y^{3/2}e^{-y/2}}{2\sqrt{2\pi}}\cdot\int_0^{\pi/2}(1-\cos^2\theta)\sin\theta\,d\theta \quad (CV)\ u := \cos\theta$$

$$f_Y(y) \overset{CV}{=} \frac{y^{3/2}e^{-y/2}}{2\sqrt{2\pi}}\cdot\int_0^1(1-u^2)\,du = \frac{y^{3/2}e^{-y/2}}{2\sqrt{2\pi}}\cdot\left[u-\frac{u^3}{3}\right]_{u=0}^{u=1}$$

$$\overset{FTC}{=} \frac{y^{3/2}e^{-y/2}}{2\sqrt{2\pi}}\cdot\left[\left(1-\frac{1^3}{3}\right)-\left(0-\frac{0^3}{3}\right)\right] = \frac{y^{3/2}e^{-y/2}}{2\sqrt{2\pi}}\cdot\frac{2}{3} = \frac{y^{3/2}e^{-y/2}}{3\sqrt{2\pi}} = \frac{y^{5/2-1}e^{-y/2}}{2^{5/2}\cdot\Gamma(5/2)} \quad \square$$

χ^2_6 from χ^2_5 & χ^2_1 pdf's

Theorem

(χ^2_6 Theorem – CHISQ6THM)

$X_1, X_2 \overset{IND}{\sim} \chi^2_1, \chi^2_5 \implies Y := X_1 + X_2 \sim \chi^2_6$ *where* $f_Y(y) = \frac{1}{16} y^2 e^{-y/2} = \frac{y^{6/2-1} e^{-y/2}}{2^{6/2} \cdot \Gamma(6/2)} \leftarrow \Gamma_{3,2}$ *pdf*

This distribution of Y is called the ***chi-square distribution with six degrees of freedom****.*

<u>PROOF:</u>

pdf's $f_{X_1}(x_1) = \frac{x_1^{-1/2} e^{-x_1/2}}{\sqrt{2\pi}}, \; f_{X_2}(x_2) = \frac{x_2^{3/2} e^{-x_2/2}}{3\sqrt{2\pi}} \implies$ Supports $\Omega_{X_1} = \Omega_{X_2} = (0, \infty)$

Joint pdf $f_{X_1,X_2}(x_1, x_2) \overset{IND}{=} f_{X_1}(x_1) \cdot f_{X_2}(x_2) = \frac{x_1^{-1/2} e^{-x_1/2}}{\sqrt{2\pi}} \cdot \frac{x_2^{3/2} e^{-x_2/2}}{3\sqrt{2\pi}} = \frac{x_1^{-1/2} x_2^{3/2} e^{-(x_1+x_2)/2}}{6\pi}$

Joint Support $\Omega_{X_1,X_2} \overset{IND}{=} \Omega_{X_1} \times \Omega_{X_2} = \{(x_1, x_2) : x_1 \in \Omega_{X_1}, x_2 \in \Omega_{X_2}\} = \{(x_1, x_2) : x_1 > 0, x_2 > 0\}$

Change of random variables (CRV):

$$\begin{matrix} Y := X_1 + X_2 \\ W := X_2 \end{matrix} \iff \begin{matrix} X_1 = Y - W \\ X_2 = W \end{matrix} \implies J = \begin{bmatrix} \frac{\partial X_1}{\partial Y} & \frac{\partial X_1}{\partial W} \\ \frac{\partial X_2}{\partial Y} & \frac{\partial X_2}{\partial W} \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \implies |J| = 1$$

$\implies$ Joint pdf $f_{Y,W}(y, w) \overset{CRV}{=} f_{X_1,X_2}(y - w, w) \cdot |J| = \frac{(y-w)^{-1/2} w^{3/2} e^{-y/2}}{6\pi}$

$\implies$ Joint Support $\Omega_{Y,W} = \{(y, w) : 0 < y < \infty, \; 0 < w < y\}$

χ_6^2 from χ_5^2 & χ_1^2 pdf's

Theorem

(χ_6^2 Theorem – CHISQ6THM)

$X_1, X_2 \overset{IND}{\sim} \chi_1^2, \chi_5^2 \implies Y := X_1 + X_2 \sim \chi_6^2$ *where* $f_Y(y) = \frac{1}{16} y^2 e^{-y/2} = \frac{y^{6/2-1} e^{-y/2}}{2^{6/2} \cdot \Gamma(6/2)} \leftarrow \Gamma_{3,2}$ *pdf*

*This distribution of Y is called the **chi-square distribution with six degrees of freedom**.*

PROOF:

$\implies$ Joint pdf $f_{Y,W}(y,w) \overset{CRV}{=} f_{X_1,X_2}(y-w, w) \cdot |J| = \dfrac{(y-w)^{-1/2} w^{3/2} e^{-y/2}}{6\pi}$

$\implies$ Joint Support $\Omega_{Y,W} = \{(y,w) : 0 < y < \infty,\ 0 < w < y\}$

$\implies$ Marginal pdf $f_Y(y) = \displaystyle\int_{\underline{\partial}\Omega_1^V}^{\overline{\partial}\Omega_1^V} f_{Y,W}(y,w)\, dw = \int_0^y \frac{(y-w)^{-1/2} w^{3/2} e^{-y/2}}{6\pi}\, dw$

$$f_Y(y) = \frac{e^{-y/2}}{6\pi} \int_0^y \frac{w^{3/2}\, dw}{(y-w)^{1/2}} \quad (CV)\ w := y \sin^2\theta \iff \begin{cases} y - w & = & y\cos^2\theta \\ dw & = & 2y \sin\theta \cos\theta\ d\theta \\ w = y & \iff & \theta = \pi/2 \\ w = 0 & \iff & \theta = 0 \end{cases}$$

$$f_Y(y) \overset{CV}{=} \frac{e^{-y/2}}{6\pi} \cdot \int_0^{\pi/2} \frac{(y^{3/2}\sin^3\theta)(2y\sin\theta\cos\theta\ d\theta)}{y^{1/2}\cos\theta}$$

$$f_Y(y) = \frac{y^2 e^{-y/2}}{3\pi} \cdot \int_0^{\pi/2} \sin^4\theta\ d\theta \overset{TRIG}{=} \frac{y^2 e^{-y/2}}{3\pi} \cdot \int_0^{\pi/2} \left(\frac{1}{2} - \frac{1}{2}\cos 2\theta\right)^2\ d\theta$$

χ^2_6 from χ^2_5 & χ^2_1 pdf's

Theorem

(χ^2_6 Theorem – CHISQ6THM)

$X_1, X_2 \overset{IND}{\sim} \chi^2_1, \chi^2_5 \implies Y := X_1 + X_2 \sim \chi^2_6$ *where* $f_Y(y) = \frac{1}{16} y^2 e^{-y/2} = \frac{y^{6/2-1} e^{-y/2}}{2^{6/2} \cdot \Gamma(6/2)} \leftarrow \Gamma_{3,2}$ *pdf*

*This distribution of Y is called the **chi-square distribution with six degrees of freedom**.*

PROOF:

$\implies$ Joint pdf $f_{Y,W}(y,w) \overset{CRV}{=} f_{X_1,X_2}(y-w, w) \cdot |J| = \dfrac{(y-w)^{-1/2} w^{3/2} e^{-y/2}}{6\pi}$

$\implies$ Joint Support $\Omega_{Y,W} = \{(y,w) : 0 < y < \infty,\ 0 < w < y\}$

$\implies$ Marginal pdf $f_Y(y) = \displaystyle\int_{\underline{\partial}\Omega_1^V}^{\overline{\partial}\Omega_1^V} f_{Y,W}(y,w)\, dw = \int_0^y \frac{(y-w)^{-1/2} w^{3/2} e^{-y/2}}{6\pi}\, dw$

$$f_Y(y) = \frac{y^2 e^{-y/2}}{3\pi} \cdot \int_0^{\pi/2} \left(\frac{1}{2} - \frac{1}{2}\cos 2\theta\right)^2 d\theta = \frac{y^2 e^{-y/2}}{3\pi} \cdot \int_0^{\pi/2} \left(\frac{1}{4} - \frac{1}{2}\cos 2\theta + \frac{1}{4}\cos^2 2\theta\right) d\theta$$

$$f_Y(y) \overset{TRIG}{=} \frac{y^2 e^{-y/2}}{3\pi} \cdot \int_0^{\pi/2} \left[\frac{1}{4} - \frac{1}{2}\cos 2\theta + \frac{1}{4}\left(\frac{1}{2} + \frac{1}{2}\cos 4\theta\right)\right] d\theta$$

$$f_Y(y) = \frac{y^2 e^{-y/2}}{3\pi} \cdot \int_0^{\pi/2} \left(\frac{3}{8} - \frac{1}{2}\cos 2\theta + \frac{1}{8}\cos 4\theta\right) d\theta$$

χ^2_6 from χ^2_5 & χ^2_1 pdf's

Theorem

(χ^2_6 Theorem – CHISQ6THM)

$X_1, X_2 \overset{IND}{\sim} \chi^2_1, \chi^2_5 \implies Y := X_1 + X_2 \sim \chi^2_6$ *where* $f_Y(y) = \frac{1}{16} y^2 e^{-y/2} = \frac{y^{6/2-1} e^{-y/2}}{2^{6/2} \cdot \Gamma(6/2)} \leftarrow \Gamma_{3,2}$ *pdf*

This distribution of Y is called the ***chi-square distribution with six degrees of freedom****.*

PROOF:

$\implies$ Joint pdf $f_{Y,W}(y,w) \overset{CRV}{=} f_{X_1,X_2}(y-w, w) \cdot |J| = \dfrac{(y-w)^{-1/2} w^{3/2} e^{-y/2}}{6\pi}$

$\implies$ Joint Support $\Omega_{Y,W} = \{(y,w) : 0 < y < \infty,\ 0 < w < y\}$

$\implies$ Marginal pdf $f_Y(y) = \displaystyle\int_{\underline{\partial}\Omega_1^V}^{\overline{\partial}\Omega_1^V} f_{Y,W}(y,w)\, dw = \int_0^y \frac{(y-w)^{-1/2} w^{3/2} e^{-y/2}}{6\pi}\, dw$

$$f_Y(y) = \frac{y^2 e^{-y/2}}{3\pi} \cdot \int_0^{\pi/2} \left(\frac{3}{8} - \frac{1}{2}\cos 2\theta + \frac{1}{8}\cos 4\theta \right) d\theta$$

$$f_Y(y) = \frac{y^2 e^{-y/2}}{3\pi} \cdot \left[\frac{3}{8}\theta - \frac{1}{4}\sin 2\theta + \frac{1}{32}\sin 4\theta \right]_{\theta=0}^{\theta=\pi/2}$$

$$f_Y(y) \overset{FTC}{=} \frac{y^2 e^{-y/2}}{3\pi} \cdot \frac{3\pi}{16} = \frac{1}{16} y^2 e^{-y/2} = \frac{y^{6/2-1} e^{-y/2}}{2^{6/2} \cdot \Gamma(6/2)} \qquad \square$$

χ^2_7 from χ^2_6 & χ^2_1 pdf's

Theorem

(χ^2_7 Theorem – CHISQ7THM)

$X_1, X_2 \overset{IND}{\sim} \chi^2_1, \chi^2_6 \implies Y := X_1 + X_2 \sim \chi^2_7$ *where* $f_Y(y) = \dfrac{y^{5/2} e^{-y/2}}{15\sqrt{2\pi}} = \dfrac{y^{7/2-1} e^{-y/2}}{2^{7/2} \cdot \Gamma(7/2)} \leftarrow \Gamma_{7/2,2}$ *pdf*

*This distribution of Y is called the **chi-square distribution with seven degrees of freedom**.*

PROOF:

pdf's $f_{X_1}(x_1) = \dfrac{x_1^{-1/2} e^{-x_1/2}}{\sqrt{2\pi}}, \; f_{X_2}(x_2) = \dfrac{1}{16} x_2^2 e^{-x_2/2} \implies$ Supports $\Omega_{X_1} = \Omega_{X_2} = (0, \infty)$

Joint pdf $f_{X_1,X_2}(x_1, x_2) \overset{IND}{=} f_{X_1}(x_1) \cdot f_{X_2}(x_2) = \dfrac{x_1^{-1/2} e^{-x_1/2}}{\sqrt{2\pi}} \cdot \dfrac{1}{16} x_2^2 e^{-x_2/2} = \dfrac{x_1^{-1/2} x_2^2 e^{-(x_1+x_2)/2}}{16\sqrt{2\pi}}$

Joint Support $\Omega_{X_1,X_2} \overset{IND}{=} \Omega_{X_1} \times \Omega_{X_2} = \{(x_1, x_2) : x_1 \in \Omega_{X_1}, x_2 \in \Omega_{X_2}\} = \{(x_1, x_2) : x_1 > 0, x_2 > 0\}$

Change of random variables (CRV):

$$\begin{array}{l} Y := X_1 + X_2 \\ W := X_2 \end{array} \iff \begin{array}{l} X_1 = Y - W \\ X_2 = W \end{array} \implies J = \begin{bmatrix} \frac{\partial X_1}{\partial Y} & \frac{\partial X_1}{\partial W} \\ \frac{\partial X_2}{\partial Y} & \frac{\partial X_2}{\partial W} \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \implies |J| = 1$$

$\implies$ Joint pdf $f_{Y,W}(y, w) \overset{CRV}{=} f_{X_1,X_2}(y - w, w) \cdot |J| = \dfrac{(y - w)^{-1/2} w^2 e^{-y/2}}{16\sqrt{2\pi}}$

$\implies$ Joint Support $\Omega_{Y,W} = \{(y, w) : 0 < y < \infty, \; 0 < w < y\}$

χ^2_7 from χ^2_6 & χ^2_1 pdf's

Theorem

(χ^2_7 Theorem – CHISQ7THM)

$X_1, X_2 \overset{IND}{\sim} \chi^2_1, \chi^2_6 \implies Y := X_1 + X_2 \sim \chi^2_7$ *where* $f_Y(y) = \dfrac{y^{5/2}e^{-y/2}}{15\sqrt{2\pi}} = \dfrac{y^{7/2-1}e^{-y/2}}{2^{7/2}\cdot\Gamma(7/2)} \leftarrow \Gamma_{7/2,2}$ *pdf*

This distribution of Y is called the ***chi-square distribution with seven degrees of freedom****.*

PROOF:

$\implies$ Joint pdf $f_{Y,W}(y,w) \overset{CRV}{=} f_{X_1,X_2}(y-w,w)\cdot|J| = \dfrac{(y-w)^{-1/2}w^2e^{-y/2}}{16\sqrt{2\pi}}$

$\implies$ Joint Support $\Omega_{Y,W} = \{(y,w) : 0 < y < \infty,\ 0 < w < y\}$

$\implies$ Marginal pdf $f_Y(y) = \displaystyle\int_{\underline{\partial}\Omega_1^V}^{\overline{\partial}\Omega_1^V} f_{Y,W}(y,w)\,dw = \int_0^y \frac{(y-w)^{-1/2}w^2e^{-y/2}}{16\sqrt{2\pi}}\,dw$

$$f_Y(y) = \frac{e^{-y/2}}{16\sqrt{2\pi}}\int_0^y \frac{w^2\,dw}{(y-w)^{1/2}} \quad (CV)\ w := y\sin^2\theta \iff \begin{cases} y-w & = & y\cos^2\theta \\ dw & = & 2y\sin\theta\cos\theta\,d\theta \\ w = y & \iff & \theta = \pi/2 \\ w = 0 & \iff & \theta = 0 \end{cases}$$

$$f_Y(y) \overset{CV}{=} \frac{e^{-y/2}}{16\sqrt{2\pi}}\cdot\int_0^{\pi/2}\frac{(y^2\sin^4\theta)(2y\sin\theta\cos\theta\,d\theta)}{y^{1/2}\cos\theta}$$

$$f_Y(y) = \frac{y^{5/2}e^{-y/2}}{8\sqrt{2\pi}}\cdot\int_0^{\pi/2}\sin^5\theta\,d\theta \overset{TRIG}{=} \frac{y^{5/2}e^{-y/2}}{8\sqrt{2\pi}}\cdot\int_0^{\pi/2}\left(\frac{1}{2}-\frac{1}{2}\cos 2\theta\right)^2\sin\theta\,d\theta$$

χ_7^2 from χ_6^2 & χ_1^2 pdf's

Theorem

(χ_7^2 Theorem – CHISQ7THM)

$X_1, X_2 \overset{IND}{\sim} \chi_1^2, \chi_6^2 \implies Y := X_1 + X_2 \sim \chi_7^2$ *where* $f_Y(y) = \dfrac{y^{5/2}e^{-y/2}}{15\sqrt{2\pi}} = \dfrac{y^{7/2-1}e^{-y/2}}{2^{7/2}\cdot\Gamma(7/2)} \leftarrow \Gamma_{7/2,2}$ *pdf*

*This distribution of Y is called the **chi-square distribution with seven degrees of freedom**.*

PROOF:

$\implies$ Joint pdf $f_{Y,W}(y,w) \overset{CRV}{=} f_{X_1,X_2}(y-w,w)\cdot|J| = \dfrac{(y-w)^{-1/2}w^2e^{-y/2}}{16\sqrt{2\pi}}$

$\implies$ Joint Support $\Omega_{Y,W} = \{(y,w) : 0 < y < \infty,\ 0 < w < y\}$

$\implies$ Marginal pdf $f_Y(y) = \displaystyle\int_{\underline{\partial}\Omega_1^V}^{\overline{\partial}\Omega_1^V} f_{Y,W}(y,w)\,dw = \int_0^y \frac{(y-w)^{-1/2}w^2e^{-y/2}}{16\sqrt{2\pi}}\,dw$

$$f_Y(y) = \frac{y^{5/2}e^{-y/2}}{8\sqrt{2\pi}}\cdot\int_0^{\pi/2}\left(\frac{1}{2}-\frac{1}{2}\cos 2\theta\right)^2\sin\theta\,d\theta$$

$$f_Y(y) \overset{TRIG}{=} \frac{y^{5/2}e^{-y/2}}{8\sqrt{2\pi}}\cdot\int_0^{\pi/2}\left[\frac{1}{2}-\frac{1}{2}\left(2\cos^2\theta-1\right)\right]^2\sin\theta\,d\theta$$

$$f_Y(y) = \frac{y^{5/2}e^{-y/2}}{8\sqrt{2\pi}}\cdot\int_0^{\pi/2}\left(1-\cos^2\theta\right)^2\sin\theta\,d\theta$$

χ^2_7 from χ^2_6 & χ^2_1 pdf's

Theorem

(χ^2_7 Theorem – CHISQ7THM)

$X_1, X_2 \overset{IND}{\sim} \chi^2_1, \chi^2_6 \implies Y := X_1 + X_2 \sim \chi^2_7$ *where* $f_Y(y) = \dfrac{y^{5/2}e^{-y/2}}{15\sqrt{2\pi}} = \dfrac{y^{7/2-1}e^{-y/2}}{2^{7/2}\cdot\Gamma(7/2)} \leftarrow \Gamma_{7/2,2}$ *pdf*

This distribution of Y is called the ***chi-square distribution with seven degrees of freedom****.*

PROOF:

$\implies$ Joint pdf $f_{Y,W}(y,w) \overset{CRV}{=} f_{X_1,X_2}(y-w,w)\cdot|J| = \dfrac{(y-w)^{-1/2}w^2e^{-y/2}}{16\sqrt{2\pi}}$

$\implies$ Joint Support $\Omega_{Y,W} = \{(y,w) : 0 < y < \infty,\ 0 < w < y\}$

$\implies$ Marginal pdf $f_Y(y) = \displaystyle\int_{\underline{\partial}\Omega_1^V}^{\overline{\partial}\Omega_1^V} f_{Y,W}(y,w)\,dw = \int_0^y \frac{(y-w)^{-1/2}w^2e^{-y/2}}{16\sqrt{2\pi}}\,dw$

$$f_Y(y) = \frac{y^{5/2}e^{-y/2}}{8\sqrt{2\pi}}\cdot\int_0^{\pi/2}\left(1-\cos^2\theta\right)^2\sin\theta\,d\theta = \frac{y^{5/2}e^{-y/2}}{8\sqrt{2\pi}}\cdot\int_0^{\pi/2}\left(1-2\cos^2\theta+\cos^4\theta\right)\sin\theta\,d\theta$$

$(CV)\ u := \cos\theta \iff du = -\sin\theta \implies u(\pi/2) = \cos(\pi/2) = 0,\ u(0) = \cos(0) = 1$

$$f_Y(y) \overset{CV}{=} \frac{y^{5/2}e^{-y/2}}{8\sqrt{2\pi}}\cdot\int_0^1\left(1-2u^2+u^4\right)du = \frac{y^{5/2}e^{-y/2}}{8\sqrt{2\pi}}\cdot\left[u-\frac{2}{3}u^3+\frac{1}{5}u^5\right]_{u=0}^{u=1}$$

$$f_Y(y) \overset{FTC}{=} \frac{y^{5/2}e^{-y/2}}{8\sqrt{2\pi}}\cdot\frac{8}{15} = \frac{y^{5/2}e^{-y/2}}{15\sqrt{2\pi}} = \frac{y^{7/2-1}e^{-y/2}}{2^{7/2}\cdot\Gamma(7/2)} \qquad \square$$

χ^2_{2k} Pattern

CHISQ RV	PDF SIMPLIFIED	PDF GAMMA FORM	KEY INTEGRAL
$Y \sim \chi^2_2$	$\frac{1}{2}e^{-y/2}$	$\frac{y^{2/2-1}e^{-y/2}}{2^{2/2}\cdot\Gamma(2/2)}$	$\int_0^{\pi/2} \sin^0\theta\, d\theta = \frac{\pi}{2}$
$Y \sim \chi^2_4$	$\frac{1}{4}ye^{-y/2}$	$\frac{y^{4/2-1}e^{-y/2}}{2^{4/2}\cdot\Gamma(4/2)}$	$\int_0^{\pi/2} \sin^2\theta\, d\theta = \frac{\pi}{4}$
$Y \sim \chi^2_6$	$\frac{1}{16}y^2e^{-y/2}$	$\frac{y^{6/2-1}e^{-y/2}}{2^{6/2}\cdot\Gamma(6/2)}$	$\int_0^{\pi/2} \sin^4\theta\, d\theta = \frac{3\pi}{16}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

$$\int_0^{\pi/2} \sin^{2k}\theta\, d\theta = \frac{\sqrt{\pi}}{2}\cdot\frac{\Gamma\left(k+\frac{1}{2}\right)}{\Gamma\left(k+1\right)} = \frac{(1)(3)(5)\cdots(2k-1)}{(2)(4)(6)\cdots(2k)}\cdot\frac{\pi}{2}, \quad k\in\mathbb{Z}_+$$

$$\int_0^{\pi/2} \sin^{2k-2}\theta\, d\theta = \frac{\sqrt{\pi}}{2}\cdot\frac{\Gamma\left(k-\frac{1}{2}\right)}{\Gamma\left(k\right)} = \frac{(1)(3)(5)\cdots(2k-3)}{(2)(4)(6)\cdots(2k-2)}\cdot\frac{\pi}{2}, \quad k\in\mathbb{Z}_+$$

H.B. Dwight, *Tables and Integrals and Other Math. Data*, 4^{th} Ed, 1961. (Ch12, entry 858.44)
B.O. Peirce, R.M. Foster, *A Short Table of Integrals*, 4^{th} Ed, 1956. (Part II, entry 498)

χ^2_{2k+1} Pattern

CHISQ RV	PDF SIMPLIFIED	PDF GAMMA FORM	KEY INTEGRAL
$Y \sim \chi^2_3$	$\frac{1}{\sqrt{2\pi}} y^{1/2} e^{-y/2}$	$\frac{y^{3/2-1} e^{-y/2}}{2^{3/2} \cdot \Gamma(3/2)}$	$\int_0^{\pi/2} \sin\theta \, d\theta = 1$
$Y \sim \chi^2_5$	$\frac{1}{3\sqrt{2\pi}} y^{3/2} e^{-y/2}$	$\frac{y^{5/2-1} e^{-y/2}}{2^{5/2} \cdot \Gamma(5/2)}$	$\int_0^{\pi/2} \sin^3\theta \, d\theta = \frac{2}{3}$
$Y \sim \chi^2_7$	$\frac{1}{15\sqrt{2\pi}} y^{5/2} e^{-y/2}$	$\frac{y^{7/2-1} e^{-y/2}}{2^{7/2} \cdot \Gamma(7/2)}$	$\int_0^{\pi/2} \sin^5\theta \, d\theta = \frac{8}{15}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

$$\int_0^{\pi/2} \sin^{2k+1}\theta \, d\theta = \frac{\sqrt{\pi}}{2} \cdot \frac{\Gamma(k+1)}{\Gamma\left((k+1)+\frac{1}{2}\right)} = \frac{(2)(4)(6)\cdots(2k)}{(3)(5)(7)\cdots(2k+1)}, \quad k \in \mathbb{Z}_+$$

$$\int_0^{\pi/2} \sin^{2k-1}\theta \, d\theta = \frac{\sqrt{\pi}}{2} \cdot \frac{\Gamma(k)}{\Gamma\left(k+\frac{1}{2}\right)} = \frac{(2)(4)(6)\cdots(2k-2)}{(3)(5)(7)\cdots(2k-1)}, \quad k \in \mathbb{Z}_+$$

H.B. Dwight, *Tables and Integrals and Other Math. Data*, 4^{th} Ed, 1961. (Ch12, entry 858.44)
B.O. Peirce, R.M. Foster, *A Short Table of Integrals*, 4^{th} Ed, 1956. (Part II, entry 498)

χ^2_{2k} from χ^2_{2k-1} & χ^2_1 pdf's

Theorem

(χ^2 Even Theorem – CHISQETHM)

$X_1, X_2 \overset{IND}{\sim} \chi^2_1, \chi^2_{2k-1} \implies Y := X_1 + X_2 \sim \chi^2_{2k}$ *where* $f_Y(y) = \dfrac{y^{(2k)/2-1}e^{-y/2}}{2^{(2k)/2} \cdot \Gamma((2k)/2)}$

*This distribution of Y is called the **chi-square distribution with** $(2k)$ **degrees of freedom**.*

<u>PROOF:</u>

pdf's $f_{X_1}(x_1) = \dfrac{x_1^{-1/2}e^{-x_1/2}}{2^{1/2} \cdot \Gamma(1/2)}$, $f_{X_2}(x_2) = \dfrac{x_2^{k-3/2}e^{-x_2/2}}{2^{k-1/2} \cdot \Gamma(k-1/2)} \implies$ Supports $\Omega_{X_1} = \Omega_{X_2} = (0, \infty)$

Joint pdf $f_{X_1,X_2}(x_1, x_2) \overset{IND}{=} f_{X_1}(x_1) \cdot f_{X_2}(x_2) = \dfrac{x_1^{-1/2}x_2^{k-3/2}e^{-(x_1+x_2)/2}}{2^{k-1/2} \cdot 2^{1/2} \cdot \Gamma(k-1/2) \cdot \Gamma(1/2)}$

Joint Support $\Omega_{X_1,X_2} \overset{IND}{=} \Omega_{X_1} \times \Omega_{X_2} = \{(x_1, x_2) : x_1 \in \Omega_{X_1}, x_2 \in \Omega_{X_2}\} = \{(x_1, x_2) : x_1 > 0, x_2 > 0\}$

Change of random variables (CRV):

$$\begin{array}{l} Y := X_1 + X_2 \\ W := X_2 \end{array} \iff \begin{array}{l} X_1 = Y - W \\ X_2 = W \end{array} \implies J = \begin{bmatrix} \frac{\partial X_1}{\partial Y} & \frac{\partial X_1}{\partial W} \\ \frac{\partial X_2}{\partial Y} & \frac{\partial X_2}{\partial W} \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \implies |J| = 1$$

$\implies$ Joint pdf $f_{Y,W}(y, w) \overset{CRV}{=} f_{X_1,X_2}(y-w, w) \cdot |J| = \dfrac{(y-w)^{-1/2}w^{k-3/2}e^{-y/2}}{2^{k-1/2} \cdot 2^{1/2} \cdot \Gamma(k-1/2) \cdot \Gamma(1/2)}$

$\implies$ Joint Support $\Omega_{Y,W} = \{(y, w) : 0 < y < \infty,\ 0 < w < y\}$

χ^2_{2k} from χ^2_{2k-1} & χ^2_1 pdf's

Theorem

(χ^2 Even Theorem – CHISQETHM)

$X_1, X_2 \overset{IND}{\sim} \chi^2_1, \chi^2_{2k-1} \implies Y := X_1 + X_2 \sim \chi^2_{2k}$ *where* $f_Y(y) = \dfrac{y^{(2k)/2-1} e^{-y/2}}{2^{(2k)/2} \cdot \Gamma((2k)/2)}$

*This distribution of Y is called the **chi-square distribution with** $(2k)$ **degrees of freedom**.*

PROOF:

$\implies$ Joint pdf $f_{Y,W}(y,w) \overset{CRV}{=} f_{X_1,X_2}(y-w, w) \cdot |J| = \dfrac{x_1^{-1/2} x_2^{k-3/2} e^{-(x_1+x_2)/2}}{2^{k-1/2} \cdot 2^{1/2} \cdot \Gamma(k-1/2) \cdot \Gamma(1/2)}$

$\implies$ Joint Support $\Omega_{Y,W} = \{(y,w) : 0 < y < \infty,\ 0 < w < y\}$

$\implies$ Marginal pdf $f_Y(y) = \displaystyle\int_{\underline{\partial}\Omega_1^V}^{\overline{\partial}\Omega_1^V} f_{Y,W}(y,w)\, dw = \int_0^y \frac{(y-w)^{-1/2} w^{k-3/2} e^{-y/2}}{2^{k-1/2} \cdot 2^{1/2} \cdot \Gamma(k-1/2) \cdot \Gamma(1/2)}\, dw$

$$f_Y(y) = \int_0^y \frac{(y-w)^{-1/2} w^{k-3/2} e^{-y/2}}{2^{k-1/2} \cdot 2^{1/2} \cdot \Gamma(k-1/2) \cdot \Gamma(1/2)}\, dw \ (CV)\ w := y\sin^2\theta \Leftrightarrow \begin{cases} y - w = y\cos^2\theta \\ dw = 2y\sin\theta\cos\theta\ d\theta \\ w = y \iff \theta = \pi/2 \\ w = 0 \iff \theta = 0 \end{cases}$$

$$f_Y(y) \overset{CV}{=} \frac{1}{2^{k-1/2} \cdot 2^{1/2} \cdot \Gamma(k-1/2) \cdot \Gamma(1/2)} \cdot \int_0^{\pi/2} \frac{y^{k-3/2} e^{-y/2} \sin^{2k-3}\theta \cdot (2y\sin\theta\cos\theta\ d\theta)}{y^{1/2}\cos\theta}$$

χ^2_{2k} from χ^2_{2k-1} & χ^2_1 pdf's

Theorem

(χ^2 Even Theorem – CHISQETHM)

$X_1, X_2 \overset{IND}{\sim} \chi^2_1, \chi^2_{2k-1} \implies Y := X_1 + X_2 \sim \chi^2_{2k}$ *where* $f_Y(y) = \dfrac{y^{(2k)/2-1}e^{-y/2}}{2^{(2k)/2} \cdot \Gamma((2k)/2)}$

*This distribution of Y is called the **chi-square distribution with** $(2k)$ **degrees of freedom**.*

PROOF:

$\implies$ Joint pdf $f_{Y,W}(y,w) \overset{CRV}{=} f_{X_1,X_2}(y-w,w) \cdot |J| = \dfrac{x_1^{-1/2} x_2^{k-3/2} e^{-(x_1+x_2)/2}}{2^{k-1/2} \cdot 2^{1/2} \cdot \Gamma(k-1/2) \cdot \Gamma(1/2)}$

$\implies$ Joint Support $\Omega_{Y,W} = \{(y,w) : 0 < y < \infty,\ 0 < w < y\}$

$\implies$ Marginal pdf $f_Y(y) = \displaystyle\int_{\underline{\partial}\Omega_1^V}^{\overline{\partial}\Omega_1^V} f_{Y,W}(y,w)\, dw = \int_0^y \frac{(y-w)^{-1/2} w^{k-3/2} e^{-y/2}}{2^{k-1/2} \cdot 2^{1/2} \cdot \Gamma(k-1/2) \cdot \Gamma(1/2)}\, dw$

$$f_Y(y) \overset{CV}{=} \frac{1}{2^{k-1/2} \cdot 2^{1/2} \cdot \Gamma(k-1/2) \cdot \Gamma(1/2)} \cdot \int_0^{\pi/2} \frac{y^{k-3/2} e^{-y/2} \sin^{2k-3}\theta \cdot (2y \sin\theta \cos\theta\, d\theta)}{y^{1/2} \cos\theta}$$

$$f_Y(y) = \frac{y^{k-1} e^{-y/2}}{2^{k-3/2} \cdot 2^{1/2} \cdot \Gamma(k-1/2) \cdot \Gamma(1/2)} \cdot \int_0^{\pi/2} \sin^{2k-2}\theta\, d\theta$$

$$f_Y(y) = \frac{y^{k-1} e^{-y/2}}{2^{k-3/2} \cdot 2^{1/2} \cdot \Gamma(k-1/2) \cdot \Gamma(1/2)} \cdot \left[\frac{\sqrt{\pi}}{2} \cdot \frac{\Gamma(k-1/2)}{\Gamma(k)}\right] = \frac{y^{(2k)/2-1} e^{-y/2}}{2^{(2k)/2} \cdot \Gamma((2k)/2)} \qquad \square$$

χ^2_{2k+1} from χ^2_{2k} & χ^2_1 pdf's

Theorem

(χ^2 Odd Theorem – CHISQOTHM)

$X_1, X_2 \overset{IND}{\sim} \chi^2_1, \chi^2_{2k} \implies Y := X_1 + X_2 \sim \chi^2_{2k+1}$ *where* $f_Y(y) = \dfrac{y^{(2k+1)/2-1} e^{-y/2}}{2^{(2k+1)/2} \cdot \Gamma((2k+1)/2)}$

This distribution of Y is called the ***chi-square distribution with*** $(2k+1)$ ***degrees of freedom***.

<u>PROOF:</u>

pdf's $f_{X_1}(x_1) = \dfrac{x_1^{-1/2} e^{-x_1/2}}{2^{1/2} \cdot \Gamma(1/2)}, \; f_{X_2}(x_2) = \dfrac{x_2^{k-1} e^{-x_2/2}}{2^k \cdot \Gamma(k)} \implies$ Supports $\Omega_{X_1} = \Omega_{X_2} = (0, \infty)$

Joint pdf $f_{X_1,X_2}(x_1, x_2) \overset{IND}{=} f_{X_1}(x_1) \cdot f_{X_2}(x_2) = \dfrac{x_1^{-1/2} x_2^{k-1} e^{-(x_1+x_2)/2}}{2^k \cdot 2^{1/2} \cdot \Gamma(k) \cdot \Gamma(1/2)}$

Joint Support $\Omega_{X_1,X_2} \overset{IND}{=} \Omega_{X_1} \times \Omega_{X_2} = \{(x_1, x_2) : x_1 \in \Omega_{X_1}, x_2 \in \Omega_{X_2}\} = \{(x_1, x_2) : x_1 > 0, x_2 > 0\}$

Change of random variables (CRV):

$$\begin{array}{l} Y := X_1 + X_2 \\ W := X_2 \end{array} \iff \begin{array}{l} X_1 = Y - W \\ X_2 = W \end{array} \implies J = \begin{bmatrix} \frac{\partial X_1}{\partial Y} & \frac{\partial X_1}{\partial W} \\ \frac{\partial X_2}{\partial Y} & \frac{\partial X_2}{\partial W} \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \implies |J| = 1$$

$\implies$ Joint pdf $f_{Y,W}(y, w) \overset{CRV}{=} f_{X_1,X_2}(y - w, w) \cdot |J| = \dfrac{(y-w)^{-1/2} w^{k-1} e^{-y/2}}{2^k \cdot 2^{1/2} \cdot \Gamma(k) \cdot \Gamma(1/2)}$

$\implies$ Joint Support $\Omega_{Y,W} = \{(y, w) : 0 < y < \infty, \; 0 < w < y\}$

χ^2_{2k+1} from χ^2_{2k} & χ^2_1 pdf's

Theorem

(χ^2 Odd Theorem – CHISQOTHM)

$X_1, X_2 \overset{IND}{\sim} \chi^2_1, \chi^2_{2k} \implies Y := X_1 + X_2 \sim \chi^2_{2k+1}$ *where* $f_Y(y) = \dfrac{y^{(2k+1)/2-1} e^{-y/2}}{2^{(2k+1)/2} \cdot \Gamma((2k+1)/2)}$

This distribution of Y is called the ***chi-square distribution with*** $(2k+1)$ ***degrees of freedom***.

PROOF:

$\implies$ Joint pdf $f_{Y,W}(y,w) \overset{CRV}{=} f_{X_1,X_2}(y-w, w) \cdot |J| = \dfrac{x_1^{-1/2} x_2^{k-1} e^{-(x_1+x_2)/2}}{2^k \cdot 2^{1/2} \cdot \Gamma(k) \cdot \Gamma(1/2)}$

$\implies$ Joint Support $\Omega_{Y,W} = \{(y,w) : 0 < y < \infty,\ 0 < w < y\}$

$\implies$ Marginal pdf $f_Y(y) = \displaystyle\int_{\underline{\partial}\Omega_1^V}^{\overline{\partial}\Omega_1^V} f_{Y,W}(y,w)\, dw = \int_0^y \frac{(y-w)^{-1/2} w^{k-1} e^{-y/2}}{2^k \cdot 2^{1/2} \cdot \Gamma(k) \cdot \Gamma(1/2)}\, dw$

$$f_Y(y) = \int_0^y \frac{(y-w)^{-1/2} w^{k-1} e^{-y/2}}{2^k \cdot 2^{1/2} \cdot \Gamma(k) \cdot \Gamma(1/2)}\, dw \quad (CV)\ w := y\sin^2\theta \iff \begin{cases} y - w & = y\cos^2\theta \\ dw & = 2y\sin\theta\cos\theta\, d\theta \\ w = y & \iff \theta = \pi/2 \\ w = 0 & \iff \theta = 0 \end{cases}$$

$$f_Y(y) \overset{CV}{=} \frac{1}{2^k \cdot 2^{1/2} \cdot \Gamma(k) \cdot \Gamma(1/2)} \cdot \int_0^{\pi/2} \frac{y^{k-1} \sin^{2k-2}\theta \cdot (2y\sin\theta\cos\theta\, d\theta)}{y^{1/2}\cos\theta}$$

χ^2_{2k+1} from χ^2_{2k} & χ^2_1 pdf's

Theorem

(χ^2 Odd Theorem – CHISQOTHM)

$X_1, X_2 \overset{IND}{\sim} \chi^2_1, \chi^2_{2k} \implies Y := X_1 + X_2 \sim \chi^2_{2k+1}$ *where* $f_Y(y) = \dfrac{y^{(2k+1)/2-1} e^{-y/2}}{2^{(2k+1)/2} \cdot \Gamma((2k+1)/2)}$

*This distribution of Y is called the **chi-square distribution with** $(2k+1)$ **degrees of freedom**.*

PROOF:

$\implies$ Joint pdf $f_{Y,W}(y,w) \overset{CRV}{=} f_{X_1,X_2}(y-w,w) \cdot |J| = \dfrac{x_1^{-1/2} x_2^{k-1} e^{-(x_1+x_2)/2}}{2^k \cdot 2^{1/2} \cdot \Gamma(k) \cdot \Gamma(1/2)}$

$\implies$ Joint Support $\Omega_{Y,W} = \{(y,w) : 0 < y < \infty,\ 0 < w < y\}$

$\implies$ Marginal pdf $f_Y(y) = \displaystyle\int_{\underline{\partial}\Omega_1^V}^{\overline{\partial}\Omega_1^V} f_{Y,W}(y,w)\, dw = \int_0^y \frac{(y-w)^{-1/2} w^{k-1} e^{-y/2}}{2^k \cdot 2^{1/2} \cdot \Gamma(k) \cdot \Gamma(1/2)}\, dw$

$$f_Y(y) \overset{CV}{=} \frac{1}{2^k \cdot 2^{1/2} \cdot \Gamma(k) \cdot \Gamma(1/2)} \cdot \int_0^{\pi/2} \frac{y^{k-1} \sin^{2k-2}\theta \cdot (2y \sin\theta \cos\theta\, d\theta)}{y^{1/2} \cos\theta}$$

$$f_Y(y) = \frac{y^{k-1/2} e^{-y/2}}{2^{k-1} \cdot 2^{1/2} \cdot \Gamma(k) \cdot \Gamma(1/2)} \cdot \int_0^{\pi/2} \sin^{2k-1}\theta\, d\theta$$

$$f_Y(y) = \frac{y^{k-1/2} e^{-y/2}}{2^{k-1} \cdot 2^{1/2} \cdot \Gamma(k) \cdot \Gamma(1/2)} \cdot \left[\frac{\sqrt{\pi}}{2} \cdot \frac{\Gamma(k)}{\Gamma\left(k+\frac{1}{2}\right)}\right]$$

χ^2_{2k+1} from χ^2_{2k} & χ^2_1 pdf's

Theorem

(χ^2 Odd Theorem – CHISQOTHM)

$X_1, X_2 \overset{IND}{\sim} \chi^2_1, \chi^2_{2k} \implies Y := X_1 + X_2 \sim \chi^2_{2k+1}$ *where* $f_Y(y) = \dfrac{y^{(2k+1)/2-1}e^{-y/2}}{2^{(2k+1)/2} \cdot \Gamma((2k+1)/2)}$

*This distribution of Y is called the **chi-square distribution with** $(2k+1)$ **degrees of freedom**.*

PROOF:

$\implies$ Joint pdf $f_{Y,W}(y,w) \overset{CRV}{=} f_{X_1,X_2}(y-w,w) \cdot |J| = \dfrac{x_1^{-1/2}x_2^{k-1}e^{-(x_1+x_2)/2}}{2^k \cdot 2^{1/2} \cdot \Gamma(k) \cdot \Gamma(1/2)}$

$\implies$ Joint Support $\Omega_{Y,W} = \{(y,w) : 0 < y < \infty,\ 0 < w < y\}$

$\implies$ Marginal pdf $f_Y(y) = \displaystyle\int_{\underline{\partial}\Omega_1^V}^{\overline{\partial}\Omega_1^V} f_{Y,W}(y,w)\,dw = \int_0^y \frac{(y-w)^{-1/2}w^{k-1}e^{-y/2}}{2^k \cdot 2^{1/2} \cdot \Gamma(k) \cdot \Gamma(1/2)}\,dw$

$$f_Y(y) = \frac{y^{k-1/2}e^{-y/2}}{2^{k-1} \cdot 2^{1/2} \cdot \Gamma(k) \cdot \Gamma(1/2)} \cdot \left[\frac{\sqrt{\pi}}{2} \cdot \frac{\Gamma(k)}{\Gamma\left(k+\frac{1}{2}\right)}\right]$$

$$f_Y(y) = \frac{y^{k-1/2}e^{-y/2}}{2^k \cdot 2^{1/2} \cdot \Gamma\left(k+\frac{1}{2}\right)} = \frac{y^{(2k+1)/2-1}e^{-y/2}}{2^{(2k+1)/2} \cdot \Gamma\left((2k+1)/2\right)} \qquad \square$$

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