Motivation: Gamma Function
Engineering Statistics

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PART I:

Gamma Function

Definition
Convergence
Properties

Gamma Function (Definition)

There are many definitions of the **gamma function**, but we will only consider Euler's improper integral definition devised circa 1730:

Definition

(Gamma Function Integral Definition – GFID)

$$\Gamma(\alpha) := \int_0^\infty x^{\alpha-1} e^{-x} \, dx, \quad \text{where } \alpha > 0$$

Gamma Function (Convergence)

Since the gamma function is an improper integral,
it's convergence must be formally proven:

Theorem

(Gamma Function Convergence Theorem – GFCT)

The improper integral $I := \displaystyle\int_0^\infty x^{\alpha-1} e^{-x} \, dx$ *converges for* $\alpha > 0$

Gamma Function (Convergence)

Theorem

(Gamma Function Convergence Theorem – GFCT)

$$\text{The improper integral } \quad I := \int_0^\infty x^{\alpha-1}e^{-x}\,dx \quad \text{converges for } \alpha > 0$$

<u>PROOF:</u> Fix $\alpha > 0$ and split integral as $I = \int_0^1 x^{\alpha-1}e^{-x}\,dx + \int_1^\infty x^{\alpha-1}e^{-x}\,dx \equiv I_0 + I_\infty$

Consider integral $I_\epsilon := \int_\epsilon^1 x^{\alpha-1}e^{-x}\,dx$ where $0 < \epsilon \ll 1$. Then, $e^{-x} < 1 \;\forall x \in (0,1]$

$$\implies I_\epsilon := \int_\epsilon^1 x^{\alpha-1}e^{-x}\,dx < \int_\epsilon^1 x^{\alpha-1}\,dx = \left[\frac{x^\alpha}{\alpha}\right]_{x=\epsilon}^{x=1} \overset{FTC}{=} \frac{1}{\alpha} - \frac{\epsilon^\alpha}{\alpha} < \frac{1}{\alpha} < \infty \implies I_\epsilon \text{ is bounded}$$

Since, for fixed $\alpha > 0$, the integrand of I_ϵ, $x^{\alpha-1}e^{-x}$ is always positive for all $x \in (0,1]$,
decreasing the value of ϵ towards zero (with fixed $\alpha > 0$) results in I_ϵ monotonically increasing.

Therefore, since integral I_ϵ is bounded above and monotically increases as $\epsilon \downarrow 0$,

$$\lim_{\epsilon\downarrow 0} I_\epsilon = I_0 \equiv \int_0^1 x^{\alpha-1}e^{-x}\,dx \quad \text{converges for all } \alpha > 0.$$

Gamma Function (Convergence)

Theorem

(Gamma Function Convergence Theorem – GFCT)

The improper integral $\;I := \int_0^\infty x^{\alpha-1} e^{-x}\, dx\;$ converges for $\;\alpha > 0$

<u>PROOF:</u> Fix $\alpha > 0$ and split integral as $I = \int_0^1 x^{\alpha-1} e^{-x}\, dx + \int_1^\infty x^{\alpha-1} e^{-x}\, dx \equiv I_0 + I_\infty$

Consider integral $I_\omega := \int_1^\omega x^{\alpha-1} e^{-x}\, dx$ where $1 \ll \omega < \infty$. Recall: $e^x \overset{TS}{=} \sum_{k=0}^\infty \frac{x^k}{k!}\;\; \forall x \in \mathbb{R}$

$\implies e^x > \dfrac{x^K}{K!}\;\; \forall K \geq 0 \implies$ in particular, $e^x > \dfrac{x^{\alpha+1}}{(\alpha+1)!} \implies e^{-x} = \dfrac{1}{e^x} < \dfrac{(\alpha+1)!}{x^{\alpha+1}}\;\; \forall x \in [1, \infty)$

$\implies x^{\alpha-1} e^{-x} < \dfrac{(\alpha+1)!}{x^{\alpha+1+(1-\alpha)}} = \dfrac{(\alpha+1)!}{x^2}\;\; \forall x \in [1, \infty)$

$I_\omega < \int_1^\omega \dfrac{(\alpha+1)!}{x^2}\, dx = \left[-\dfrac{(\alpha+1)!}{x} \right]_{x=1}^{x=\omega} \overset{FTC}{=} (\alpha+1)! \cdot \left(1 - \dfrac{1}{\omega} \right) < (\alpha+1)! \implies I_\omega$ is bounded

Since, for fixed $\alpha > 0$, the integrand of I_ω, $x^{\alpha-1} e^{-x}$ is always positive for all $x \in [1, \infty)$,
increasing the value of ω towards infinity (with fixed $\alpha > 0$) results in I_ω monotonically increasing.

Therefore, since integral I_ω is bounded above and monotically increases as $\omega \uparrow \infty$,

$$\lim_{\omega \uparrow \infty} I_\omega = I_\infty \equiv \int_1^\infty x^{\alpha-1} e^{-x}\, dx \;\; \text{converges for all } \alpha > 0.$$

Gamma Function (Convergence)

Theorem

(Gamma Function Convergence Theorem – GFCT)

$$\text{The improper integral} \quad I := \int_0^\infty x^{\alpha-1}e^{-x}\,dx \quad \text{converges for} \ \alpha > 0$$

<u>PROOF:</u> Fix $\alpha > 0$ and split integral as $I = \int_0^1 x^{\alpha-1}e^{-x}\,dx + \int_1^\infty x^{\alpha-1}e^{-x}\,dx \equiv I_0 + I_\infty$

Finally, since integrals I_0 and I_∞ have each been shown to converge for all $\alpha > 0$,

their sum $I := \int_0^\infty x^{\alpha-1}e^{-x}\,dx$ also converges for all $\alpha > 0$. $\qquad \square$

E. Artin, *The Gamma Function* / *Einführung in die Theorie der Gammafunktion*, 1964/1931. (Ch2)

Gamma Function (Properties)

$$\Gamma(\alpha) := \int_0^\infty x^{\alpha-1} e^{-x} \, dx, \quad \text{where } \alpha > 0$$

$$
\begin{array}{rcll}
\Gamma(1) & = & 1 & \\
\Gamma(\alpha) & = & (\alpha - 1) \cdot \Gamma(\alpha - 1) & \text{where } \alpha > 1 \\
\Gamma(n) & = & (n-1)! & \text{where } n \in \mathbb{Z}_+
\end{array}
$$

$$
\begin{array}{rcll}
\Gamma\left(\tfrac{1}{2}\right) & = & \sqrt{\pi} & \\
\Gamma\left(n + \tfrac{1}{2}\right) & = & \left(n - \tfrac{1}{2}\right) \cdot \Gamma\left(n - \tfrac{1}{2}\right) & \text{where } n \in \mathbb{Z}_+
\end{array}
$$

$$\Gamma\left(n + \tfrac{1}{2}\right) = \frac{\sqrt{\pi}}{2^n} \cdot [(1)(3)(5)\cdots(2n-5)(2n-3)(2n-1)] \quad \text{where } n \in \mathbb{Z}_+$$

$$\Gamma\left(n + \tfrac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2n-1}} \cdot \frac{\Gamma(2n)}{\Gamma(n)} \quad \text{where } n \in \mathbb{Z}_+$$

E. Artin, *The Gamma Function* / *Einführung in die Theorie der Gammafunktion*, 1964/1931. (Ch2)

H.B. Dwight, *Tables and Integrals and Other Math. Data*, 4th Ed, 1961. (Ch12, entry 850.8)

B.O. Peirce, R.M. Foster, *A Short Table of Integrals*, 4th Ed, 1956. (Part III, entry 791)

L.A. Pipes, *Applied Math. for Engineers & Physicists*, McGraw-Hill, 1946. (Ch XII, §4)

Gamma Function (Properties)

Lemma

(Gamma Function of One – GF1)

$$\Gamma(1) = 1$$

<u>PROOF:</u> $\Gamma(1) = \displaystyle\int_0^\infty e^{-x}\,dx = \Big[-e^{-x}\Big]_{x=0}^{x\to\infty} \overset{FTC}{=} -\lim_{x\to\infty} e^{-x} - (-e^0) = -0 - (-1) = 1$ $\square$

Gamma Function (Properties)

Lemma

(Gamma Function Reduction Formula – GFRF)

$$\Gamma(\alpha) = (\alpha - 1) \cdot \Gamma(\alpha - 1) \quad \text{where } \alpha > 1$$

<u>PROOF:</u> Apply Integration by Parts.

(IBP): Let $\begin{cases} u &=& x^{\alpha-1} \\ dv &=& e^{-x}\,dx \end{cases} \iff \begin{cases} du &=& (\alpha-1)x^{\alpha-2}\,dx \\ v &=& -e^{-x} \end{cases}$

$$
\begin{aligned}
\implies \Gamma(\alpha) &= \int_0^\infty x^{\alpha-1} e^{-x}\,dx \equiv \int_0^\infty u\,dv \stackrel{IBP}{=} \Big[u(x)v(x) \Big]_{x=0}^{x\to\infty} - \int_0^\infty v\,du \\[2mm]
&= \Big[-x^{\alpha-1}e^{-x} \Big]_{x=0}^{x\to\infty} - \int_0^\infty \left(-e^{-x} \right) \cdot (\alpha-1)x^{\alpha-2}\,dx \\[2mm]
&\stackrel{FTC}{=} \left[-\lim_{x\to\infty}\left(\frac{x^{\alpha-1}}{e^x} \right) - (-0) \right] + (\alpha-1) \cdot \int_0^\infty x^{\alpha-2}e^{-x}\,dx \\[2mm]
&\stackrel{(*)}{=} 0 + (\alpha-1) \cdot \int_0^\infty x^{(\alpha-1)-1}e^{-x}\,dx \qquad (*) \text{ L'Hospital's Rule or Tower of Power} \\[2mm]
&\stackrel{\Gamma}{=} (\alpha-1) \cdot \Gamma(\alpha-1) \quad \square
\end{aligned}
$$

Gamma Function (Properties)

Lemma

(Gamma Function of Half of Unity – GF½) $\qquad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

PROOF: Apply two appropriate Changes of Variables leading to Standard Normal pdf.

(CV1): Let $x = y^2 \iff dx = 2y\,dy \implies \begin{cases} x = 0 & \iff & y = \sqrt{0} = 0 \\ x = \infty & \iff & y = \sqrt{\infty} = \infty \end{cases}$

$$\implies \Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}\,dx \stackrel{CV1}{=} \int_0^\infty y^{2(\alpha-1)}e^{-y^2}\cdot(2y\,dy) = 2\cdot\int_0^\infty y^{2\alpha-1}e^{-y^2}\,dy$$

(CV2): Let $y = \dfrac{z}{\sqrt{2}} \iff y^2 = \dfrac{z^2}{2} \iff dy = \dfrac{1}{\sqrt{2}}\,dz \implies \begin{cases} y = 0 & \iff & z = 0 \\ y = \infty & \iff & z = \infty \end{cases}$

$$\Gamma\left(\frac{1}{2}\right) \quad \stackrel{CV1}{=} \quad 2\cdot\int_0^\infty e^{-y^2}\,dy \quad \stackrel{CV2}{=} \quad 2\cdot\int_0^\infty \frac{1}{\sqrt{2}}e^{-z^2/2}\,dz \quad \stackrel{CIO}{=} \quad 2\sqrt{\pi}\cdot\int_0^\infty \underbrace{\frac{1}{\sqrt{2\pi}}e^{-z^2/2}}_{N(0,1)\ \text{pdf}}\,dz$$

$$\stackrel{N(0,1)}{=} \quad 2\sqrt{\pi}\cdot\frac{1}{2} \quad (\text{Since } N(0,1) \text{ is symmetric about zero})$$

$$\therefore\ \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \quad \square$$

L.A. Pipes, *Applied Mathematics for Engineers & Physicists*, McGraw-Hill, 1946. (Ch XII, §4)

Fin

Fin.