Hamilton's Apportionment Method

Contemporary Math

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Apportionment (Historic U.S. Definition)

Article I, Section 2 of the U.S. Constitution:

"Representatives and direct taxes shall be **apportioned** among the several states which may be included within the Union, according to their respective numbers."

Definition

Apportionment is the determination of the number of House of Representative **seats** for each **state**, provided each state's seat count is **proportional** to its population.

Apportionment (General Definition)

Realize that apportionment applies to situations other than seats of Congress.

Definition

(Apportionment)

Apportionment is the determination of the number of **identical gifts** for each **recipient**, provided some **proportionality criterion** is satisfied.

(**GIFTS**) are given to (**RECIPIENTS**) based on (**PROPORTIONALITY CR**).

GIFTS	RECIPIENTS	PROPORTIONALITY CRITERION
seats	states	(state) population
council seats	unions	(union) membership
teachers	campuses	(campus) enrollments
nurses	hospital shifts	avg # patients (per shift)
buses	bus routes	# of riders (per route)
2-hour sections	class subjects	student interest (in subject)
cookies	children	chore completion (per child)

Why are Apportionment Methods Necessary??

We know that some **GIFTS** such as pizzas and drinks can **always** be divided among recipients according to **any** proportionality criterion **perfectly**.

However, most **GIFTS** are **indivisible**, meaning having a **fractional part** is **impractical**, **useless**, or **forbidden**:

- it's impractical to have a **fraction** of a **bus**
- it's useless to have a **fraction** of a **seat**
- it's forbidden to have a **fraction** of a **2-hour section**

A Naïve Apportionment Method (Example)

WEX 10-1-1: Apportion 13 seats to 3 states based on population. (see below)

STATE:	State 1	State 2	State 3
POPULATION:	124	97	109

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STATE:	State 1	State 2	State 3
POPULATION:	124	97	109

$$P = \sum_{k=1}^{3} P_k = P_1 + P_2 + P_3 = 124 + 97 + 109 = 330$$

Step 1: Compute the **Total Population**.

A Naïve Apportionment Method (Example)

WEX 10-1-1: Apportion 13 seats to 3 states based on population. (see below)

STATE:	State 1	State 2	State 3
POPULATION:	124	97	109
FAIR SHARE:	$124/330$	$97/330$	$109/330$

$$P = \sum_{k=1}^{3} P_k = P_1 + P_2 + P_3 = 124 + 97 + 109 = 330$$

$$(State's\ Fair\ Share) = (State's\ Population)/(Total\ Population)$$

Step 1: Compute the **Total Population**.
Step 2: Compute each state's **Fair Share**.

A Naïve Apportionment Method (Example)

WEX 10-1-1: Apportion 13 seats to 3 states based on population. (see below)

STATE:	State 1	State 2	State 3
POPULATION:	124	97	109
FAIR SHARE:	37.58%	29.39%	33.03%

$$P = \sum_{k=1}^{3} P_k = P_1 + P_2 + P_3 = 124 + 97 + 109 = 330$$

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Step 1: Compute the **Total Population**.
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STATE:	State 1	State 2	State 3
POPULATION:	124	97	109
FAIR SHARE:	37.58%	29.39%	33.03%
APPORTIONMENT:	$(0.3758)(13)$	$(0.2939)(13)$	$(0.3303)(13)$

$$P = \sum_{k=1}^{3} P_k = P_1 + P_2 + P_3 = 124 + 97 + 109 = 330$$

$$(State's\ Fair\ Share) = (State's\ Population)/(Total\ Population)$$

$$(State's\ Apportionment) = (State's\ Fair\ Share) \times (Number\ of\ Seats)$$

Step 1: Compute the **Total Population.**
Step 2: Compute each state's **Fair Share.**
Step 3: Compute each state's **Apportionment.**

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STATE:	State 1	State 2	State 3
POPULATION:	124	97	109
FAIR SHARE:	37.58%	29.39%	33.03%
APPORTIONMENT:	4.8854	3.8207	4.2939

$$P = \sum_{k=1}^{3} P_k = P_1 + P_2 + P_3 = 124 + 97 + 109 = 330$$

$$(State's\ Fair\ Share) = (State's\ Population)/(Total\ Population)$$

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Step 1: Compute the **Total Population**.
Step 2: Compute each state's **Fair Share**.
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$$(State's\ Fair\ Share) = (State's\ Population)/(Total\ Population)$$

$$(State's\ Apportionment) = (State's\ Fair\ Share) \times (Number\ of\ Seats)$$

Therefore:
State 1 gets 4.8854 seats
State 2 gets 3.8207 seats
State 3 gets 4.2939 seats

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$$P = \sum_{k=1}^{3} P_k = P_1 + P_2 + P_3 = 124 + 97 + 109 = 330$$

$$(State's\ Fair\ Share) = (State's\ Population)/(Total\ Population)$$

$$(State's\ Apportionment) = (State's\ Fair\ Share) \times (Number\ of\ Seats)$$

Therefore:
State 1 gets 4.8854 seats
State 2 gets 3.8207 seats
State 3 gets 4.2939 seats
But each state receives a **fractional part** of a seat!!!!

Divisors & Quotas in the Context of Apportionment

(State's **Fair Share**) $= (State's\ Population)/(Total\ Population)$

$$
\begin{aligned}
(\text{State's } \textbf{Fair Quota}) &= (State's\ Fair\ Share) \times (Number\ of\ Seats) \\
&= \frac{(State's\ Population)}{(Total\ Population)} \times (Number\ of\ Seats) \\
&= (State's\ Population) \times \frac{(Number\ of\ Seats)}{(Total\ Population)} \\
&= (State's\ Population) \div \frac{(Total\ Population)}{(Number\ of\ Seats)} \\
&= \frac{(Total\ Population)}{(Standard\ Divisor)}
\end{aligned}
$$

Definition

$$\textbf{Standard Divisor} = \frac{(Total\ Population)}{(Number\ of\ Seats)}$$

Characteristics of Useful Apportionment Methods

So, the previous example shows why apportionment methods are necessary.

But as the naïve method shown earlier indicates, a proper apportionment method requires the following:

- A ”suitable” **divisor**
- A ”suitable” **rounding scheme**

Ideally, an apportionment method should also satisfy the **Quota Rule**.

Proposition

(Quota Rule)

*No state shall be apportioned seats beyond its **fair share quota**, rounded up or down.*

Rounding Numbers (Compact Notation)

It is often convenient to have mathematical notation for **rounding numbers.**

Always Round Down: $\lfloor 3 \rfloor = 3$ $\lfloor 3.1 \rfloor = 3$ $\lfloor 3.5 \rfloor = 3$ $\lfloor 3.9 \rfloor = 3$

Always Round Up: $\lceil 3 \rceil = 3$ $\lceil 3.1 \rceil = 4$ $\lceil 3.5 \rceil = 4$ $\lceil 3.9 \rceil = 4$

Round to Nearest Integer: $[\![3]\!] = 3$ $[\![3.1]\!] = 3$ $[\![3.5]\!] = 4$ $[\![3.9]\!] = 4$

$\lfloor x \rfloor$ is called the **floor function.**

$\lceil x \rceil$ is called the **ceiling function.**

Hamilton's Method (of Apportionment)

Proposition

(Hamilton's Method)

*Given N states w/ populations $P_1, P_2, P_3, \ldots, P_{N-1}, P_N$, and **total population** P*
Given M seats to be apportioned among the N states
Determine the apportionment for each state, labeled $A_1, A_2, A_3, \ldots, A_{N-1}, A_N$

STEP 1:	*Compute **standard divisor***	$D = \dfrac{P}{M}$
STEP 2:	*Compute quotas, **always rounding down***	$Q_k = \left\lfloor \dfrac{P_k}{D} \right\rfloor$
STEP 3:	*Compute the **total quota***	$T = \sum_{k=1}^{N} Q_k$
STEP 4:	*If $T < M$, assign each of the $(M - T)$ surplus seats (one at a time) to the states having **quotas** with the **largest fractional parts***	$A_k = Q_k + (surplus)$

Hamilton's Method (Example)

WEX 10-1-2: Apportion 13 seats to 3 states using Hamilton's Method.

STATE:	State 1	State 2	State 3
POPULATION:	124	97	109

Hamilton's Method (Example)

<u>WEX 10-1-2:</u> Apportion 13 seats to 3 states using Hamilton's Method.

STATE:	State 1	State 2	State 3
POPULATION:	124	97	109

$$N = 3, M = 13, P = \sum_{k=1}^{3} P_k = P_1 + P_2 + P_3 = 124 + 97 + 109 = 330$$

STEP 0: Collect given information.

Hamilton's Method (Example)

WEX 10-1-2: Apportion 13 seats to 3 states using Hamilton's Method.

STATE:	State 1	State 2	State 3
POPULATION:	124	97	109

$$N = 3, M = 13, P = 330$$

STEP 0: Collect given information.

<u>WEX 10-1-2:</u> Apportion 13 seats to 3 states using Hamilton's Method.

STATE:	State 1	State 2	State 3
POPULATION:	124	97	109

$$N = 3, M = 13, P = 330, D = \frac{P}{M} = \frac{330}{13} = 25.3846$$

STEP 1: Compute **standard divisor**.

Hamilton's Method (Example)

WEX 10-1-2: Apportion 13 seats to 3 states using Hamilton's Method.

STATE:	State 1	State 2	State 3
POPULATION:	124	97	109
FAIR QUOTA:	$124/D$	$97/D$	$109/D$

$$N = 3, M = 13, P = 330, D = \frac{P}{M} = \frac{330}{13} = 25.3846$$

STEP 2: Compute quotas, **rounding down**: $Q_k = \left\lfloor \frac{P_k}{D} \right\rfloor$

Hamilton's Method (Example)

WEX 10-1-2: Apportion 13 seats to 3 states using Hamilton's Method.

STATE:	State 1	State 2	State 3
POPULATION:	124	97	109
FAIR QUOTA:	4.8849	3.8212	4.2939

$$N = 3, M = 13, P = 330, D = 25.3846$$

STEP 2: Compute quotas, **rounding down**:

$$Q_k = \left\lfloor \frac{P_k}{D} \right\rfloor$$

Hamilton's Method (Example)

WEX 10-1-2: Apportion 13 seats to 3 states using Hamilton's Method.

STATE:	State 1	State 2	State 3
POPULATION:	124	97	109
FAIR QUOTA:	4.8849	3.8212	4.2939
QUOTA:	4	3	4

$$N = 3, M = 13, P = 330, D = 25.3846$$

STEP 2: Compute quotas, **rounding down**:

$$Q_k = \left\lfloor \frac{P_k}{D} \right\rfloor$$

Hamilton's Method (Example)

WEX 10-1-2: Apportion 13 seats to 3 states using Hamilton's Method.

STATE:	State 1	State 2	State 3
POPULATION:	124	97	109
FAIR QUOTA:	4.8849	3.8212	4.2939
QUOTA:	4	3	4

$$N = 3, M = 13, P = 330, D = 25.3846, T = \sum_{k=1}^{3} Q_k = 4 + 3 + 4 = 11$$

STEP 3: Compute the **total quota**.

Hamilton's Method (Example)

WEX 10-1-2: Apportion 13 seats to 3 states using Hamilton's Method.

STATE:	State 1	State 2	State 3
POPULATION:	124	97	109
FAIR QUOTA:	4.8849	3.8212	4.2939
QUOTA:	4	3	4

$$N = 3, M = 13, P = 330, D = 25.3846, T = 11$$

STEP 3: Compute the **total quota**.

Hamilton's Method (Example)

WEX 10-1-2: Apportion 13 seats to 3 states using Hamilton's Method.

STATE:	State 1	State 2	State 3
POPULATION:	124	97	109
FAIR QUOTA:	4.8849	3.8212	4.2939
QUOTA:	4	3	4
APPORTIONMENT:	4+1	3+1	4

$$N = 3, M = 13, P = 330, D = 25.3846, T = 11$$

STEP 4: Since $T < M$, add a **unit surplus** to the $(M - T) = 2$ quotas with the **largest fractional parts**.

Hamilton's Method (Example)

WEX 10-1-2: Apportion 13 seats to 3 states using Hamilton's Method.

STATE:	State 1	State 2	State 3
POPULATION:	124	97	109
FAIR QUOTA:	4.8849	3.8212	4.2939
QUOTA:	4	3	4
APPORTIONMENT:	5	4	4

$$N = 3, M = 13, P = 330, D = 25.3846, T = 11$$

STEP 4: Since $T < M$, add a **unit surplus** to the $(M - T) = 2$ quotas with the **largest fractional parts**.

Hamilton's Method (Example)

WEX 10-1-2: Apportion 13 seats to 3 states using Hamilton's Method.

STATE:	State 1	State 2	State 3
POPULATION:	124	97	109
FAIR QUOTA:	4.8849	3.8212	4.2939
QUOTA:	4	3	4
APPORTIONMENT:	**5**	**4**	**4**

$$N = 3, M = 13, P = 330, D = 25.3846, T = 11$$

Therefore:
State 1 gets 5 seats
State 2 gets 4 seats
State 3 gets 4 seats

Degree of Representation for a State

Definition

(Average Constituency)
The **average constituency** of a state is

$$C_{state} = \frac{P_{state}}{A_{state}} \equiv \frac{(Population\ of\ State)}{(Apportionment\ for\ State)}$$

The **larger** the **avg constituency**, the **more poorly represented** the state.

A ”perfect” apportionment causes **equal** avg constituencies for **all** states.
Such a situation means **each citizen’s vote is equal to all other votes**.
Unfortunately, such an idealistic apportionment is rarely possible.

Paradox (Definition)

Definition

(Paradox)

A **paradox** is a statement that contradicts itself and yet may be true.

WARNING: Do not confuse a paradox with **irony**, which is a literary device.

Paradoxes occur in various fields of study:

PARADOX	**STATEMENT**
Socrates'	"I know that I know nothing at all."
Olbers'	Why is the night sky black if there is an infinity of stars?
Faraday's	Diluted HNO_3 corrodes steel – concentrated HNO_3 does not.
of Pesticide	Applying pesticide to a pest may increase it's abundance.
of Value	Water is more useful than diamonds, yet is a lot cheaper.

http://en.wikipedia.org/wiki/List_of_paradoxes

Paradoxes Resulting from Hamilton's Method

It turns out Hamilton's Method is subject to three possible **paradoxes**:

- Alabama Paradox
- Population Paradox
- New States Paradox

Alabama Paradox

Definition

(Alabama Paradox)

Increasing the total seats may **decrease** a state's apportionment.

This happened in 1880, when the U.S. Census Bureau discovered that Alabama would get 8 of 299 House seats, but only 7 of 300 House seats.

Alabama Paradox (Example)

<u>WEX 10-1-3:</u> Use Hamilton's Method with 11 seats & 12 seats below.
Explain (in <u>one</u> sentence) why the Alabama Paradox occurs.

STATE:	State 1	State 2	State 3
POPULATION:	531	733	1136
FAIR QUOTA: $(M = 11)$			
QUOTA: $(M = 11)$			
APPORTIONMENT: $(M = 11)$			
FAIR QUOTA: $(M = 12)$			
QUOTA: $(M = 12)$			
APPORTIONMENT: $(M = 12)$			

Alabama Paradox (Example)

<u>WEX 10-1-3:</u> Use Hamilton's Method with 11 seats & 12 seats below.
Explain (in <u>one</u> sentence) why the Alabama Paradox occurs.

STATE:	State 1	State 2	State 3
POPULATION:	531	733	1136
FAIR QUOTA: $(M = 11)$	2.4337	3.3596	5.2067
QUOTA: $(M = 11)$	2	3	5
APPORTIONMENT: $(M = 11)$			
FAIR QUOTA: $(M = 12)$			
QUOTA: $(M = 12)$			
APPORTIONMENT: $(M = 12)$			

$$N = 3, M = 11, P = 2400, D = \frac{P}{M} = 218.1818, T = \sum_{k=1}^{3} Q_k = 2 + 3 + 5 = 10$$

Alabama Paradox (Example)

WEX 10-1-3: Use Hamilton's Method with 11 seats & 12 seats below.
Explain (in one sentence) why the Alabama Paradox occurs.

STATE:	State 1	State 2	State 3
POPULATION:	531	733	1136
FAIR QUOTA: $(M = 11)$	2.4337	3.3596	5.2067
QUOTA: $(M = 11)$	2	3	5
APPORTIONMENT: $(M = 11)$	3	3	5
FAIR QUOTA: $(M = 12)$			
QUOTA: $(M = 12)$			
APPORTIONMENT: $(M = 12)$			

$$N = 3, M = 11, P = 2400, D = \frac{P}{M} = 218.1818, T = \sum_{k=1}^{3} Q_k = 2 + 3 + 5 = 10$$

Alabama Paradox (Example)

WEX 10-1-3: Use Hamilton's Method with 11 seats & 12 seats below. Explain (in one sentence) why the Alabama Paradox occurs.

STATE:	State 1	State 2	State 3
POPULATION:	531	733	1136
FAIR QUOTA: $(M = 11)$	2.4337	3.3596	5.2067
QUOTA: $(M = 11)$	2	3	5
APPORTIONMENT: $(M = 11)$	3	3	5
FAIR QUOTA: $(M = 12)$	2.6550	3.6650	5.6800
QUOTA: $(M = 12)$	2	3	5
APPORTIONMENT: $(M = 12)$			

$$N = 3, M = 12, P = 2400, D = \frac{P}{M} = 200, T = \sum_{k=1}^{3} Q_k = 2 + 3 + 5 = 10$$

Alabama Paradox (Example)

WEX 10-1-3: Use Hamilton's Method with 11 seats & 12 seats below.
Explain (in one sentence) why the Alabama Paradox occurs.

STATE:	State 1	State 2	State 3
POPULATION:	531	733	1136
FAIR QUOTA: $(M = 11)$	2.4337	3.3596	5.2067
QUOTA: $(M = 11)$	2	3	5
APPORTIONMENT: $(M = 11)$	3	3	5
FAIR QUOTA: $(M = 12)$	2.6550	3.6650	5.6800
QUOTA: $(M = 12)$	2	3	5
APPORTIONMENT: $(M = 12)$	2	4	6

$$N = 3, M = 12, P = 2400, D = \frac{P}{M} = 200, T = \sum_{k=1}^{3} Q_k = 2 + 3 + 5 = 10$$

Alabama Paradox (Example)

<u>**WEX 10-1-3:**</u> Use Hamilton's Method with 11 seats & 12 seats below.
Explain (in <u>one</u> sentence) why the Alabama Paradox occurs.

STATE:	State 1	State 2	State 3
POPULATION:	531	733	1136
APPORTIONMENT: $(M = 11)$	3	3	5
APPORTIONMENT: $(M = 12)$	2	4	6

Alabama Paradox (Example)

WEX 10-1-3: Use Hamilton's Method with 11 seats & 12 seats below. Explain (in one sentence) why the Alabama Paradox occurs.

STATE:	State 1	State 2	State 3
POPULATION:	531	733	1136
APPORTIONMENT: $(M = 11)$	3	3	5
APPORTIONMENT: $(M = 12)$	**2**	4	6

The Alabama Paradox occurs since

increasing seats from 11 to 12 caused **State 1** to **lose one seat**.

How often does the Alabama Paradox occur??

STATE:	State 1	State 2	State 3
POPULATION:	531	733	1136
APPORT. $(M = 11)$	3	3	5
APPORT. $(M = 12)$	**2**	4	6
APPORT. $(M = 47)$	11	14	22
APPORT. $(M = 48)$	**10**	15	23
APPORT. $(M = 142)$	32	43	67
APPORT. $(M = 143)$	**31**	44	68
APPORT. $(M = 178)$	40	54	84
APPORT. $(M = 179)$	**39**	55	85
APPORT. $(M = 273)$	61	83	129
APPORT. $(M = 274)$	**60**	84	130
APPORT. $(M = 368)$	82	112	174
APPORT. $(M = 369)$	**81**	113	175
APPORT. $(M = 499)$	111	152	236
APPORT. $(M = 500)$	**110**	153	237

Population Paradox

Definition

(Population Paradox)

A **small fast-growing** state can **lose** a seat to a **large slow-growing** state.

This happened in 1900, when Virginia was growing much faster than Maine, but Virginia lost a House seat & Maine gained a House seat.

THIS PARADOX OCCURS LESS FREQUENTLY THAN THE OTHER PARADOXES & IS HARDER TO DEMONSTRATE, AND HENCE WILL NO LONGER BE CONSIDERED GOING FORWARD.

New States Paradox

Definition

(New States Paradox)

The addition of a new state with its fair share of seats can affect the apportionments of other states.

This happened in 1907, when Oklahoma joined the Union.
The House increased five seats, which were to be apportioned to Oklahoma.
However, this caused Maine to gain one seat & New York to lose one seat!

New States Paradox (Example)

WEX 10-1-4: (a) Use Hamilton's Method with 30 seats on the 3 states below.

STATE:	State 1	State 2	State 3	State 4
POPULATION:	531	733	1136	**(N/A)**
FAIR QUOTA:				**(N/A)**
QUOTA:				**(N/A)**
APPORT.: (3 states)				**(N/A)**
POPULATION:	531	733	1136	324
FAIR QUOTA:				
QUOTA:				
APPORT.: (4 states)				

New States Paradox (Example)

WEX 10-1-4: (a) Use Hamilton's Method with 30 seats on the 3 states below.

STATE:	State 1	State 2	State 3	State 4
POPULATION:	531	733	1136	(N/A)
FAIR QUOTA:	6.6375	9.1625	14.2000	(N/A)
QUOTA:	6	9	14	(N/A)
APPORT.: (3 states)	7	9	14	(N/A)
POPULATION:	531	733	1136	324
FAIR QUOTA:				
QUOTA:				
APPORT.: (4 states)				

$$N = 3, M = 30, P = 2400, D = \frac{P}{M} = 80, T = \sum_{k=1}^{3} Q_k = 6 + 9 + 14 = 29$$

New States Paradox (Example)

WEX 10-1-4: (b) Use Hamilton's Method with 33 seats on the 4 states below.

STATE:	State 1	State 2	State 3	State 4
POPULATION:	531	733	1136	**(N/A)**
FAIR QUOTA:	6.6375	9.1625	14.2000	**(N/A)**
QUOTA:	6	9	14	**(N/A)**
APPORT.: (3 states)	7	9	14	**(N/A)**
POPULATION:	531	733	1136	324
FAIR QUOTA:				
QUOTA:				
APPORT.: (4 states)				

New States Paradox (Example)

WEX 10-1-4: (b) Use Hamilton's Method with 33 seats on the 4 states below.

STATE:	State 1	State 2	State 3	State 4
POPULATION:	531	733	1136	(N/A)
FAIR QUOTA:	6.6375	9.1625	14.2000	(N/A)
QUOTA:	6	9	14	(N/A)
APPORT.: (3 states)	7	9	14	(N/A)
POPULATION:	531	733	1136	324
FAIR QUOTA:	6.4328	8.8800	13.7621	3.9251
QUOTA:	6	8	13	3
APPORT.: (4 states)	6	9	14	4

$$N = 4, M = 33, P = 2400, D = \frac{P}{M} = 72.7273, T = \sum_{k=1}^{4} Q_k = 6 + 8 + 13 + 3 = 30$$

WEX 10-1-4:

(c) Explain (in one sentence) why the New States Paradox occurs.

STATE:	State 1	State 2	State 3	State 4
POPULATION:	531	733	1136	**(N/A)**
APPORT.: (3 states)	7	9	14	**(N/A)**
POPULATION:	531	733	1136	324
APPORT.: (4 states)	6	9	14	4

New States Paradox (Example)

WEX 10-1-4:

(c) Explain (in one sentence) why the New States Paradox occurs.

STATE:	State 1	State 2	State 3	State 4
POPULATION:	531	733	1136	**(N/A)**
APPORT.: (3 states)	**7**	9	14	**(N/A)**
POPULATION:	531	733	1136	324
APPORT.: (4 states)	**6**	9	14	4

The New States Paradox occurs since **including State 4** caused **State 1** to **lose one seat**.

A ”Pathological” Apportionment using Hamilton

Though rare, it's possible for Hamilton to **fail**:

Apportion $M = 5$ seats to $N = 4$ states

STATE:	State 1	State 2	State 3	State 4
POPULATION:	100	100	100	100
FAIR QUOTA:	1.25	1.25	1.25	1.25
QUOTA:	$1 + ??$	$1 + ??$	$1 + ??$	$1 + ??$

$$T = \sum_{k=1}^{4} Q_k = 1 + 1 + 1 + 1 = 4$$

Surplus Seats = $M - T = 5 - 4 = 1$

There's one surplus seat to assign, but each state has the same fair quota!!

So, which state earns the surplus seat???

Therein lies the problem!

A "Pathological" Apportionment using Hamilton

Though rare, it's possible for Hamilton to **fail**:

Apportion $M = \mathbf{7}$ seats to $N = 4$ states

STATE:	State 1	State 2	State 3	State 4
POPULATION:	100	100	100	100
FAIR QUOTA:	1.75	1.75	1.75	1.75
QUOTA:	$1 + ??$	$1 + ??$	$1 + ??$	$1 + ??$

$$T = \sum_{k=1}^{4} Q_k = 1 + 1 + 1 + 1 = 4$$

Surplus Seats = $M - T = \mathbf{7} - 4 = 3$

Three surplus seats to assign, but each state has the same fair quota!!

So, which states earn a surplus seat???

Therein lies the problem!

Fin

Fin.