Methods of Jefferson, Adams & Webster

Contemporary Math

Josh Engwer

TTU

14 July 2015

Rounding Numbers (Compact Notation)

It is often convenient to have mathematical notation for **rounding numbers.**

Always Round Down:	$\lfloor 3 \rfloor = 3$	$\lfloor 3.1 \rfloor = 3$	$\lfloor 3.5 \rfloor = 3$	$\lfloor 3.9 \rfloor = 3$
Always Round Up:	$\lceil 3 \rceil = 3$	$\lceil 3.1 \rceil = 4$	$\lceil 3.5 \rceil = 4$	$\lceil 3.9 \rceil = 4$
Round to Nearest Integer:	$[\![3]\!] = 3$	$[\![3.1]\!] = 3$	$[\![3.5]\!] = 4$	$[\![3.9]\!] = 4$

$\lfloor x \rfloor$ is called the **floor function.**

$\lceil x \rceil$ is called the **ceiling function.**

Hamilton's Method (of Apportionment)

Proposition

(Hamilton's Method)

Given N *states w/ populations* $P_1, P_2, P_3, \ldots, P_{N-1}, P_N$, *and* ***total population*** P
Given M *seats to be apportioned among the* N *states*
Determine the apportionment for each state, labeled $A_1, A_2, A_3, \ldots, A_{N-1}, A_N$

STEP 1:	*Compute **standard divisor***	$D = \dfrac{P}{M}$
STEP 2:	*Compute quotas, **rounding down***	$Q_k = \left\lfloor \dfrac{P_k}{D} \right\rfloor$
STEP 3:	*Compute the total quota*	$T = \sum_{k=1}^{N} Q_k$
STEP 4:	*If* $T < M$, *assign each of the* $(M - T)$ *surplus seats (one at a time) to the states having **quotas** with the **largest fractional parts***	$A_k = Q_k + (any\ surplus)$

Jefferson's Method (of Apportionment)

Proposition

(Jefferson's Method)

Given N states and M seats to be apportioned among the states:

STEP 1:	*Compute **standard divisor***	$D = \frac{P}{M}$
STEP 2:	*Pick **parameter** α such that*	$-1 \leq \alpha \leq 0$
STEP 3:	*Compute divisor*	$D^* = D\left[1 + \alpha\left(\frac{N}{M}\right)\right]$
STEP 4:	*Compute quotas, **rounding down***	$Q_k = \left\lfloor \frac{P_k}{D^*} \right\rfloor$
STEP 5:	*Compute the total quota*	$T = \sum_{k=1}^{N} Q_k$
STEP 6:	*If $T \neq M$, goto STEP 2* *If $T = M$, assign quotas*	$A_k = Q_k$

Adams' Method (of Apportionment)

Proposition

(Adams' Method)

Given N states and M seats to be apportioned among the states:

STEP 1: *Compute* ***standard divisor*** $D = \dfrac{P}{M}$

STEP 2: *Pick* ***parameter*** α *such that* $0 \leq \alpha \leq 1$

STEP 3: *Compute divisor* $D^* = D\left[1 + \alpha\left(\dfrac{N}{M}\right)\right]$

STEP 4: *Compute quotas,* ***rounding up*** $Q_k = \left\lceil \dfrac{P_k}{D^*} \right\rceil$

STEP 5: *Compute the total quota* $T = \displaystyle\sum_{k=1}^{N} Q_k$

STEP 6: *If $T \neq M$, goto STEP 2*
If $T = M$, assign quotas $A_k = Q_k$

Webster's Method (of Apportionment)

Proposition

(Webster's Method)

Given N states and M seats to be apportioned among the states:

STEP 1:	*Compute **standard divisor***	$D = \frac{P}{M}$
STEP 2:	*Pick **parameter** α such that*	$-1 \leq \alpha \leq 1$
STEP 3:	*Compute divisor*	$D^* = D\left[1 + \alpha\left(\frac{N}{M}\right)\right]$
STEP 4:	*Compute quotas, **rounding as usual***	$Q_k = \left[\!\left[\frac{P_k}{D^*}\right]\!\right]$
STEP 5:	*Compute the total quota*	$T = \sum_{k=1}^{N} Q_k$
STEP 6:	*If $T \neq M$, goto STEP 2* *If $T = M$, assign quotas*	$A_k = Q_k$

Choosing the Parameter α Correctly the First Time....

....is nearly impossible to do!

Therefore, α must be found by **trial & error**.

It turns out that there is a range of α-values that work.

However, the size of the range can sometimes be very tiny $(< 1/100)$

So, using trial-and-error on a range as tiny as one-hundreth in size can be long and tedious – it may take upwards of 10 guesses of α to finally find a suitable value in the range!!

Therefore, going forward with Jefferson's, Adams', and Webster's methods:

A suitable choice for α will be provided to you a priori.

Jefferson's Method (Example)

WEX 10-3-1: Using **Jefferson's Method** with $\alpha = -0.40$, apportion 13 seats to 3 states based on population below.

STATE:	State 1	State 2	State 3
POPULATION:	124	97	109

Jefferson's Method (Example)

WEX 10-3-1: Using **Jefferson's Method** with $\alpha = -0.40$, apportion 13 seats to 3 states based on population below.

STATE:	State 1	State 2	State 3
POPULATION:	124	97	109

$$\alpha = -0.40, N = 3, M = 13, P = \sum_{k=1}^{3} P_k = P_1 + P_2 + P_3 = 124 + 97 + 109 = 330$$

STEP 0: Collect given information.

Jefferson's Method (Example)

WEX 10-3-1: Using **Jefferson's Method** with $\alpha = -0.40$, apportion 13 seats to 3 states based on population below.

STATE:	State 1	State 2	State 3
POPULATION:	124	97	109

$$\alpha = -0.40, N = 3, M = 13, P = 330$$

STEP 0: Collect given information.

Jefferson's Method (Example)

WEX 10-3-1: Using **Jefferson's Method** with $\alpha = -0.40$, apportion 13 seats to 3 states based on population below.

STATE:	State 1	State 2	State 3
POPULATION:	124	97	109

$$\alpha = -0.40, N = 3, M = 13, P = 330, D = \frac{P}{M} = \frac{300}{13} = 25.3846$$

STEP 1: Compute **standard divisor**.

Jefferson's Method (Example)

WEX 10-3-1: Using **Jefferson's Method** with $\alpha = -0.40$, apportion 13 seats to 3 states based on population below.

STATE:	State 1	State 2	State 3
POPULATION:	124	97	109

$$\alpha = -0.40, N = 3, M = 13, P = 330, D = \frac{P}{M} = \frac{330}{13} = 25.3846$$

$$D^* = D\left[1 + \alpha\left(\frac{N}{M}\right)\right] = 25.3846\left[1 + (-0.40)\left(\frac{3}{13}\right)\right] = 23.0414$$

STEP 2: Compute **divisor**.

Jefferson's Method (Example)

WEX 10-3-1: Using **Jefferson's Method** with $\alpha = -0.40$, apportion 13 seats to 3 states based on population below.

STATE:	State 1	State 2	State 3
POPULATION:	124	97	109
QUOTA:	$124/D^*$	$97/D^*$	$109/D^*$

$$\alpha = -0.40, N = 3, M = 13, P = 330, D = \frac{P}{M} = \frac{330}{13} = 25.3846$$

$$D^* = D\left[1 + \alpha\left(\frac{N}{M}\right)\right] = 25.3846\left[1 + (-0.40)\left(\frac{3}{13}\right)\right] = 23.0414$$

STEP 3: Compute quotas, **rounding down**: $Q_k = \left\lfloor \frac{P_k}{D^*} \right\rfloor$

WEX 10-3-1: Using **Jefferson's Method** with $\alpha = -0.40$, apportion 13 seats to 3 states based on population below.

STATE:	State 1	State 2	State 3
POPULATION:	124	97	109
QUOTA:	5.3816	4.2098	4.7306

$$\alpha = -0.40, N = 3, M = 13, P = 330, D = \frac{P}{M} = \frac{330}{13} = 25.3846$$

$$D^* = D\left[1 + \alpha\left(\frac{N}{M}\right)\right] = 25.3846\left[1 + (-0.40)\left(\frac{3}{13}\right)\right] = 23.0414$$

STEP 3: Compute quotas, **rounding down**: $Q_k = \left\lfloor \frac{P_k}{D^*} \right\rfloor$

Jefferson's Method (Example)

WEX 10-3-1: Using **Jefferson's Method** with $\alpha = -0.40$, apportion 13 seats to 3 states based on population below.

STATE:	State 1	State 2	State 3
POPULATION:	124	97	109
QUOTA:	5	4	4

$$\alpha = -0.40, N = 3, M = 13, P = 330, D = \frac{P}{M} = \frac{330}{13} = 25.3846$$

$$D^* = D\left[1 + \alpha\left(\frac{N}{M}\right)\right] = 25.3846\left[1 + (-0.40)\left(\frac{3}{13}\right)\right] = 23.0414$$

STEP 3: Compute quotas, **rounding down**: $Q_k = \left\lfloor \frac{P_k}{D^*} \right\rfloor$

Jefferson's Method (Example)

WEX 10-3-1: Using **Jefferson's Method** with $\alpha = -0.40$, apportion 13 seats to 3 states based on population below.

STATE:	State 1	State 2	State 3
POPULATION:	124	97	109
QUOTA:	5	4	4

$$\alpha = -0.40, N = 3, M = 13, P = 330, D = \frac{P}{M} = \frac{330}{13} = 25.3846$$

$$D^* = D\left[1 + \alpha\left(\frac{N}{M}\right)\right] = 25.3846\left[1 + (-0.40)\left(\frac{3}{13}\right)\right] = 23.0414$$

Therefore:
State 1 gets 5 seats
State 2 gets 4 seats
State 3 gets 4 seats

Adams' Method (Example)

WEX 10-3-2: Using **Adams' Method** with $\alpha = 0.62$, apportion 13 seats to 3 states based on population below.

STATE:	State 1	State 2	State 3
POPULATION:	124	97	109

Adams' Method (Example)

WEX 10-3-2: Using **Adams' Method** with $\alpha = 0.62$, apportion 13 seats to 3 states based on population below.

STATE:	State 1	State 2	State 3
POPULATION:	124	97	109

$$\alpha = 0.62, N = 3, M = 13, P = \sum_{k=1}^{3} P_k = P_1 + P_2 + P_3 = 124 + 97 + 109 = 330$$

STEP 0: Collect given information.

Adams' Method (Example)

<u>**WEX 10-3-2:**</u> Using **Adams' Method** with $\alpha = 0.62$, apportion 13 seats to 3 states based on population below.

STATE:	State 1	State 2	State 3
POPULATION:	124	97	109

$$\alpha = 0.62, N = 3, M = 13, P = 330$$

STEP 0: Collect given information.

WEX 10-3-2: Using **Adams' Method** with $\alpha = 0.62$, apportion 13 seats to 3 states based on population below.

STATE:	State 1	State 2	State 3
POPULATION:	124	97	109

$$\alpha = 0.62, N = 3, M = 13, P = 330, D = \frac{P}{M} = \frac{330}{13} = 25.3846$$

STEP 1: Compute **standard divisor**.

Adams' Method (Example)

WEX 10-3-2: Using **Adams' Method** with $\alpha = 0.62$,
apportion 13 seats to 3 states based on population below.

STATE:	State 1	State 2	State 3
POPULATION:	124	97	109

$$\alpha = 0.62, N = 3, M = 13, P = 330, D = \frac{P}{M} = \frac{330}{13} = 25.3846$$

$$D^* = D\left[1 + \alpha\left(\frac{N}{M}\right)\right] = 25.3846\left[1 + (0.62)\left(\frac{3}{13}\right)\right] = 29.0166$$

STEP 2: Compute **divisor**.

Adams' Method (Example)

WEX 10-3-2: Using **Adams' Method** with $\alpha = 0.62$, apportion 13 seats to 3 states based on population below.

STATE:	State 1	State 2	State 3
POPULATION:	124	97	109
QUOTA:	$124/D^*$	$97/D^*$	$109/D^*$

$$\alpha = 0.62, N = 3, M = 13, P = 330, D = \frac{P}{M} = \frac{330}{13} = 25.3846$$

$$D^* = D\left[1 + \alpha\left(\frac{N}{M}\right)\right] = 25.3846\left[1 + (0.62)\left(\frac{3}{13}\right)\right] = 29.0166$$

STEP 3: Compute quotas, **rounding up**: $Q_k = \left\lceil \frac{P_k}{D^*} \right\rceil$

Adams' Method (Example)

<u>**WEX 10-3-2:**</u> Using **Adams' Method** with $\alpha = 0.62$, apportion 13 seats to 3 states based on population below.

STATE:	State 1	State 2	State 3
POPULATION:	124	97	109
QUOTA:	4.2734	3.3429	3.7565

$$\alpha = 0.62, N = 3, M = 13, P = 330, D = \frac{P}{M} = \frac{330}{13} = 25.3846$$

$$D^* = D\left[1 + \alpha\left(\frac{N}{M}\right)\right] = 25.3846\left[1 + (0.62)\left(\frac{3}{13}\right)\right] = 29.0166$$

STEP 3: Compute quotas, **rounding up**: $Q_k = \left\lceil \frac{P_k}{D^*} \right\rceil$

Adams' Method (Example)

WEX 10-3-2: Using **Adams' Method** with $\alpha = 0.62$, apportion 13 seats to 3 states based on population below.

STATE:	State 1	State 2	State 3
POPULATION:	124	97	109
QUOTA:	5	4	4

$$\alpha = 0.62, N = 3, M = 13, P = 330, D = \frac{P}{M} = \frac{330}{13} = 25.3846$$

$$D^* = D\left[1 + \alpha\left(\frac{N}{M}\right)\right] = 25.3846\left[1 + (0.62)\left(\frac{3}{13}\right)\right] = 29.0166$$

STEP 3: Compute quotas, **rounding up**: $Q_k = \left\lceil \frac{P_k}{D^*} \right\rceil$

Adams' Method (Example)

<u>**WEX 10-3-2:**</u> Using **Adams' Method** with $\alpha = 0.62$, apportion 13 seats to 3 states based on population below.

STATE:	State 1	State 2	State 3
POPULATION:	124	97	109
QUOTA:	5	4	4

$$\alpha = 0.62, N = 3, M = 13, P = 330, D = \frac{P}{M} = \frac{330}{13} = 25.3846$$

$$D^* = D\left[1 + \alpha\left(\frac{N}{M}\right)\right] = 25.3846\left[1 + (0.62)\left(\frac{3}{13}\right)\right] = 29.0166$$

Therefore:
State 1 gets 5 seats
State 2 gets 4 seats
State 3 gets 4 seats

Webster's Method (Example)

WEX 10-3-3: Using **Webster's Method** with $\alpha = 0$,
apportion 13 seats to 3 states based on population below.

STATE:	State 1	State 2	State 3
POPULATION:	124	97	109

Webster's Method (Example)

<u>WEX 10-3-3:</u> Using **Webster's Method** with $\alpha = 0$, apportion 13 seats to 3 states based on population below.

STATE:	State 1	State 2	State 3
POPULATION:	124	97	109

$$\alpha = 0, N = 3, M = 13, P = \sum_{k=1}^{3} P_k = P_1 + P_2 + P_3 = 124 + 97 + 109 = 330$$

STEP 0: Collect given information.

Webster's Method (Example)

WEX 10-3-3: Using **Webster's Method** with $\alpha = 0$, apportion 13 seats to 3 states based on population below.

STATE:	State 1	State 2	State 3
POPULATION:	124	97	109

$$\alpha = 0, N = 3, M = 13, P = 330$$

STEP 0: Collect given information.

Webster's Method (Example)

<u>WEX 10-3-3:</u> Using **Webster's Method** with $\alpha = 0$,
apportion 13 seats to 3 states based on population below.

STATE:	State 1	State 2	State 3
POPULATION:	124	97	109

$$\alpha = 0, N = 3, M = 13, P = 330, D = \frac{P}{M} = \frac{330}{13} = 25.3846$$

STEP 1: Compute **standard divisor**.

Webster's Method (Example)

<u>WEX 10-3-3:</u> Using **Webster's Method** with $\alpha = 0$, apportion 13 seats to 3 states based on population below.

STATE:	State 1	State 2	State 3
POPULATION:	124	97	109

$$\alpha = 0, N = 3, M = 13, P = 330, D = \frac{P}{M} = \frac{330}{13} = 25.3846$$

$$D^* = D\left[1 + \alpha\left(\frac{N}{M}\right)\right] = 25.3846\left[1 + (0)\left(\frac{3}{13}\right)\right] = 25.3846$$

STEP 2: Compute **divisor**.

Webster's Method (Example)

WEX 10-3-3: Using **Webster's Method** with $\alpha = 0$, apportion 13 seats to 3 states based on population below.

STATE:	State 1	State 2	State 3
POPULATION:	124	97	109
QUOTA:	$124/D^*$	$97/D^*$	$109/D^*$

$$\alpha = 0, N = 3, M = 13, P = 330, D = \frac{P}{M} = \frac{330}{13} = 25.3846$$

$$D^* = D\left[1 + \alpha\left(\frac{N}{M}\right)\right] = 25.3846\left[1 + (0)\left(\frac{3}{13}\right)\right] = 25.3846$$

STEP 3: Compute quotas, **rounding as usual**: $Q_k = \left[\!\left[\frac{P_k}{D^*}\right]\!\right]$

Webster's Method (Example)

WEX 10-3-3: Using **Webster's Method** with $\alpha = 0$, apportion 13 seats to 3 states based on population below.

STATE:	State 1	State 2	State 3
POPULATION:	124	97	109
QUOTA:	4.8849	3.8212	4.2939

$$\alpha = 0, N = 3, M = 13, P = 330, D = \frac{P}{M} = \frac{330}{13} = 25.3846$$

$$D^* = D\left[1 + \alpha\left(\frac{N}{M}\right)\right] = 25.3846\left[1 + (0)\left(\frac{3}{13}\right)\right] = 25.3846$$

STEP 3: Compute quotas, **rounding as usual**: $Q_k = \left[\!\left[\frac{P_k}{D^*}\right]\!\right]$

Webster's Method (Example)

WEX 10-3-3: Using **Webster's Method** with $\alpha = 0$, apportion 13 seats to 3 states based on population below.

STATE:	State 1	State 2	State 3
POPULATION:	124	97	109
QUOTA:	5	4	4

$$\alpha = 0, N = 3, M = 13, P = 330, D = \frac{P}{M} = \frac{330}{13} = 25.3846$$

$$D^* = D\left[1 + \alpha\left(\frac{N}{M}\right)\right] = 25.3846\left[1 + (0)\left(\frac{3}{13}\right)\right] = 25.3846$$

STEP 3: Compute quotas, **rounding as usual**: $Q_k = \left[\!\left[\frac{P_k}{D^*}\right]\!\right]$

Webster's Method (Example)

WEX 10-3-3: Using **Webster's Method** with $\alpha = 0$,
apportion 13 seats to 3 states based on population below.

STATE:	State 1	State 2	State 3
POPULATION:	124	97	109
QUOTA:	5	4	4

$$\alpha = 0, N = 3, M = 13, P = 330, D = \frac{P}{M} = \frac{330}{13} = 25.3846$$

$$D^* = D\left[1 + \alpha\left(\frac{N}{M}\right)\right] = 25.3846\left[1 + (0)\left(\frac{3}{13}\right)\right] = 25.3846$$

Therefore:
State 1 gets 5 seats
State 2 gets 4 seats
State 3 gets 4 seats

Different Notions of Averaging Two Numbers

Definition

(Arithmetic Mean)

The **arithmetic mean** of numbers a and b is: $\text{AM}(a, b) = \frac{1}{2}(a + b)$

Definition

(Geometric Mean)

The **geometric mean** of numbers a and b is: $\text{GM}(a, b) = \sqrt{ab}$

Definition

(Harmonic Mean)

The **harmonic mean** of numbers a and b is: $\text{HM}(a, b) = \dfrac{1}{\frac{1}{2}\left(\frac{1}{a} + \frac{1}{b}\right)}$

Different Notions of Averaging Two Numbers

a	b	AM(a,b)	GM(a,b)	HM(a,b)
1	1.25	1.125	1.118	1.111
1	1.50	1.250	1.225	1.200
1	2.00	1.500	1.414	1.333
1	3.00	2.000	1.732	1.500
1	5.00	3.000	2.236	1.667
1	10.0	5.500	3.162	1.818
1	20.0	10.50	4.472	1.905
1	50.0	25.50	7.071	1.961
1	100	50.50	10.0	1.980
1	200	100.5	14.1	1.990
1	1000	500.5	31.6	1.998

Theorem

(Inequality of Means)

$$a \leq HM(a,b) \leq GM(a,b) \leq AM(a,b) \leq b$$

Different Notions of Averaging Two Numbers

For the purposes of apportionment, these means will always involve **consecutive positive integers**:

a	b	AM(a,b)	GM(a,b)	HM(a,b)
1	2	1.500	1.414	1.333
2	3	2.500	2.449	2.400
3	4	3.500	3.464	3.429
4	5	4.500	4.472	4.444
10	11	10.500	10.488	10.476
20	21	20.500	20.494	20.488
50	51	50.500	50.4975	50.4950
100	101	100.5	100.4988	100.4975
1000	1001	1000.5	1000.49988	1000.49975
10000	10001	10000.5	10000.499988	10000.499975

Theorem

(Inequality of Means)

$$a \leq HM(a,b) \leq GM(a,b) \leq AM(a,b) \leq b$$

A Timeline of the Apportionment Methods (in the US)

TIME PERIOD	**APPORTIONMENT**	**ROUND**	**SURPLUS**
1792 to 1840	Jefferson	Down	N/A
1842	Webster	Relative to AM	N/A
1850 to 1900	Hamilton	Down	Absolute Fract. Parts
1901, 1911	Webster	Relative to AM	N/A
1921	(None Used)	N/A	N/A
1931	Webster	Relative to AM	N/A
1941-Present	**Huntington-Hill**	Relative to **GM**	N/A
(Never)	Adams	Up	N/A
(Never)	**Dean**	Relative to **HM**	N/A
(Never)	**Lowndes**	Down	Relative Fract. Parts

Computing Huntington-Hill, Dean, and Lowndes is beyond our scope.

Geometric means (GM) & harmonic means (HM) are beyond our scope.

Summary of previous Worked Examples

Apportion $M = 13$ seats to $N = 3$ states

	State 1	State 2	State 3	**Suitable α-values**
Population	124	97	109	
Hamilton	5	4	4	
Jefferson	5	4	4	$-0.61 < \alpha < -0.20$
Adams	5	4	4	$0.32 < \alpha < 0.95$
Webster	5	4	4	$-0.19 < \alpha < 0.37$
Huntington	5	4	4	$-0.17 < \alpha < 0.39$
Dean	5	4	4	$-0.14 < \alpha < 0.42$

It's possible for some of the apportionment methods to differ.

"Pathlogical" Apportionment Scenarios

Though rare, it's possible for Jefferson's Method to **fail**:

Apportion $M = 11$ seats to $N = 3$ states

	State 1	State 2	State 3
Population	120	240	360
Hamilton	2	4	5
Jefferson	(*FAILED*)	(*FAILED*)	(*FAILED*)
Adams $(\alpha = 0.5)$	2	4	5
Webster $(\alpha = 0.1)$	2	4	5
H-H $(\alpha = 0.15)$	2	4	5
Dean $(\alpha = 0.15)$	2	4	5

Why **Jefferson** fails:

$$\alpha < -\tfrac{11}{36} \implies T \geq 12 > 11 = M$$
$$\alpha \geq -\tfrac{11}{36} \implies T \leq 9 < 11 = M$$

"Pathlogical" Apportionment Scenarios

Though rare, it's possible for Adams' Method to **fail**:

Apportion $M = 13$ seats to $N = 3$ states

	State 1	State 2	State 3
Population	120	240	360
Hamilton	2	4	7
Jefferson $(\alpha = -0.5)$	2	4	7
Adams	$(FAILED)$	$(FAILED)$	$(FAILED)$
Webster $(\alpha = 0)$	2	4	7
H-H $(\alpha = -0.05)$	2	4	7
Dean $(\alpha = -0.05)$	2	4	7

Why **Adams** fails:

$$\alpha \leq \frac{13}{36} \quad \Longrightarrow \quad T \geq 15 > 13 = M$$

$$\alpha > \frac{13}{36} \quad \Longrightarrow \quad T \leq 12 < 13 = M$$

"Pathlogical" Apportionment Scenarios

Though rare, it's possible for Webster's Method to **fail**:

Apportion $M = 55$ seats to $N = 4$ states

	State 1	State 2	State 3	State 4
Population	1	9	7	19
Hamilton	1	14	11	29
Jefferson $(\alpha = -0.4)$	1	14	11	29
Adams $(\alpha = 0.7)$	2	14	11	28
Webster	*(FAILED)*	*(FAILED)*	*(FAILED)*	*(FAILED)*
H-H $(\alpha = 0.26)$	2	14	11	28
Dean $(\alpha = 0.265)$	2	14	11	28

Why **Webster** fails:

$$\alpha \leq \frac{55}{216} \implies T \geq 56 > 55 = M$$
$$\alpha > \frac{55}{216} \implies T \leq 52 < 55 = M$$

Extreme "Pathlogical" Apportionment Scenarios

Though contrived, it's possible for **all methods** to **fail**:

Apportion $M = 5$ seats to $N = 4$ states

	State 1	State 2	State 3	State 4
Population	100	100	100	100
Hamilton	$(FAILED)$	$(FAILED)$	$(FAILED)$	$(FAILED)$
Jefferson	$(FAILED)$	$(FAILED)$	$(FAILED)$	$(FAILED)$
Adams	$(FAILED)$	$(FAILED)$	$(FAILED)$	$(FAILED)$
Webster	$(FAILED)$	$(FAILED)$	$(FAILED)$	$(FAILED)$
H-H	$(FAILED)$	$(FAILED)$	$(FAILED)$	$(FAILED)$
Dean	$(FAILED)$	$(FAILED)$	$(FAILED)$	$(FAILED)$

METHOD	**THRESHOLD α-VALUES**
Jefferson	$\alpha = -\frac{15}{32} = -0.46875$
Adams	$\alpha = \frac{5}{16} = 0.3125$
Webster	$\alpha = -\frac{5}{24} \approx -0.20833$
Huntington-Hill	$\alpha \approx -0.14514$
Dean	$\alpha = -\frac{5}{64} = -0.078125$

Extreme ”Pathlogical” Apportionment Scenarios

Though contrived, it’s possible for **all methods** to **fail**:

Apportion $M = 7$ seats to $N = 4$ states

	State 1	State 2	State 3	State 4
Population	100	100	100	100
Hamilton	$(FAILED)$	$(FAILED)$	$(FAILED)$	$(FAILED)$
Jefferson	$(FAILED)$	$(FAILED)$	$(FAILED)$	$(FAILED)$
Adams	$(FAILED)$	$(FAILED)$	$(FAILED)$	$(FAILED)$
Webster	$(FAILED)$	$(FAILED)$	$(FAILED)$	$(FAILED)$
H-H	$(FAILED)$	$(FAILED)$	$(FAILED)$	$(FAILED)$
Dean	$(FAILED)$	$(FAILED)$	$(FAILED)$	$(FAILED)$

METHOD	**THRESHOLD α-VALUES**
Jefferson	$\alpha = -\frac{7}{32} = -0.21875$
Adams	$\alpha = \frac{21}{16} = 1.3125$
Webster	$\alpha = \frac{7}{24} \approx 0.29167$
Huntington-Hill	$\alpha \approx 0.4155$
Dean	$\alpha = \frac{35}{64} = 0.546875$

A Comparison of the Apportionment Methods

	HAMILTON	**JEFF.**	**ADAMS**	**WEBSTER**	**H-H**
Paradoxes?	*Yes (all 3)*	No	No	No	No
Violates Quota Rule?	No	*Yes*	*Yes*	*Yes*	*Yes*
Favors Large States	*Yes*	*Yes*	No	No	No
Favors Small States	No	No	*Yes*	No	*Yes*

Dean's Method is even more biased towards small states than H-H.

The Quest for a Perfect Apportionment Method....

....is, unfortunately, a fool's errand:

Theorem

(Young's Impossibility Theorem)

It's impossible to construct an apportionment method that:

- *Does not violate the Quota Rule*

– AND –

- *Does not produce any paradoxes*

The consensus is that **Webster's Method** is the best method most of the time.

Fin

Fin.