Weighted Voting Systems

Contemporary Math

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Weighted Voting Systems

So far, we've seen voting methods for **electing officials**.
Voting is also needed in **passing/vetoing resolutions or motions**:

Definition

(Resolution)

A **resolution** (AKA **motion**) is a "Yes or No" vote on a law or plan of action.

Definition

(Weighted Voting System)

A **weighted voting system** with N voters is described by the following:

[(quota) : (weight of voter 1), (weight of voter 2), . . . , (weight of voter N)]

$$[Q : w_1, w_2, \ldots, w_N]$$

The **quota** is the # of voters needed in this system to get a resolution passed.
The **weights** are the amount of votes controlled by voter 1, voter 2, etc...
The sum of the weights is represented by W: $W = \sum_{k=1}^{N} w_k$

Dictators & Veto Power in Weighted Voting Systems

Definition

(Dictator)

Let $[Q : w_1, w_2, \ldots, w_N]$ be a weighted voting system with N voters.
Then voter k is a **dictator** if voter k alone can pass a resolution/motion:

$$w_k \geq Q$$

Definition

(Veto Power)

Let $[Q : w_1, w_2, \ldots, w_N]$ be a weighted voting system with N voters.
Then voter k has **veto power** if voter k can prevent a motion from passing:

$$W - w_k < Q$$

Weighted Voting System (Example)

EX 11-3-1: Given the weighted voting system $[12 : 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1]$

Then: the **quota** is $Q = 12$.

Voter 1 has weight $w_1 = 1$ (i.e. Voter 1 has one vote)
Voter 2 has weight $w_2 = 1$ (i.e. Voter 2 has one vote)

$\vdots \qquad \vdots$

Voter 12 has weight $w_{12} = 1$ (i.e. Voter 12 has one vote)

The total sum of the weights is $W = \sum_{k=1}^{12} w_k = 1 + 1 + \cdots + 1 = 12$

$w_1 = 1 < 12 = Q \implies$ Voter 1 is NOT a dictator
$w_2 = 1 < 12 = Q \implies$ Voter 2 is NOT a dictator

$\vdots \qquad \vdots$

$w_{12} = 1 < 12 = Q \implies$ Voter 12 is NOT a dictator

$W - w_1 = 12 - 1 = 11 < 12 = Q \implies$ Voter 1 has veto power
$W - w_2 = 12 - 1 = 11 < 12 = Q \implies$ Voter 2 has veto power

$\vdots \qquad \vdots \qquad \vdots$

Weighted Voting System (Example)

EX 11-3-2: Given the weighted voting system $[14 : 17, 5, 3, 2]$

Then: the **quota** is $Q = 14$.

Voter 1 has weight $w_1 = 17$ (i.e. Voter 1 has 17 votes)
Voter 2 has weight $w_2 = 5$ (i.e. Voter 2 has 5 votes)
Voter 3 has weight $w_3 = 3$ (i.e. Voter 3 has 3 votes)
Voter 4 has weight $w_4 = 2$ (i.e. Voter 4 has 2 votes)

The total sum of the weights is $W = \sum_{k=1}^{4} w_k = 17 + 5 + 3 + 2 = 27$

$w_1 = 17 \geq 14 = Q \implies$ Voter 1 IS a dictator
$w_2 = 5 < 14 = Q \implies$ Voter 2 is NOT a dictator
$w_3 = 3 < 14 = Q \implies$ Voter 3 is NOT a dictator
$w_4 = 2 < 14 = Q \implies$ Voter 4 is NOT a dictator

$W - w_1 = 27 - 17 = 10 < 14 = Q \implies$ Voter 1 has veto power
$W - w_2 = 27 - 5 = 22 \geq 14 = Q \implies$ Voter 2 has NO veto power
$W - w_3 = 27 - 3 = 24 \geq 14 = Q \implies$ Voter 3 has NO veto power
$W - w_4 = 27 - 2 = 25 \geq 14 = Q \implies$ Voter 4 has NO veto power

Coalitions

To better improve the chances of passing or vetoing a resolution, some members of the voting system may form a **coalition**:

Definition

(Coalition)

A **coalition** is a group of voters who vote the same way.
A **coalition's weight** is the sum of the weights of all voters in the coalition.
If a coalition's weight $\geq$ quota, then the coalition is called a **winning coalition**.

How many different coalitions are possible??

Proposition

(Possible Coalitions)

Given a weighted voting system with N voters.

Then there are $2^N - 1$ possible coalitions formed from the N voters.

REMARK: This follows from the fact that there are 2^N subsets of a set with N elements, but one of those subsets is the empty set (which contains nothing).

Critical Voters in Winning Coalitions

However, sometimes it just takes one voter of a coalition to leave the coalition in order to render the coalition ineffective in passing or vetoing a resolution:

Definition

(Critical Voter)

A voter in a winning coalition is **critical** if that voter were to leave the coalition, then the coalition would no longer be winning.

Measuring Voting Power

How can one measure the effectiveness, or power, of a voter?
One way is the compute the **Banzhaf Power Index**:

Definition

(Banzhaf Power Index)

In a weighted voting system, a voter's **Banzhaf power index** is defined as

$$\frac{\#\text{ times }\textbf{the voter}\text{ is }\textbf{critical}\text{ in winning coalitions}}{\text{Total }\#\text{ times }\textbf{voters}\text{ are }\textbf{critical}\text{ in winning coalitions}}$$

Fin.