Probability: Unions, Intersections, Complements

Contemporary Math

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TTU

29 July 2015

Probability of an Event Not Occurring

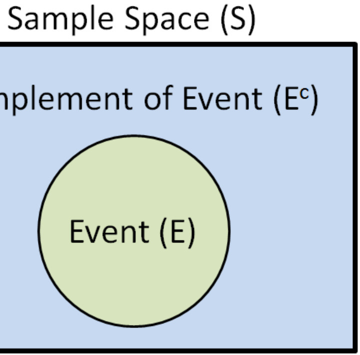
Proposition

(Probability of an Event Not Occurring)

$$P(\text{Not } E) = 1 - P(E)$$

which is equivalent to

$$P\left(E^c\right) = 1 - P(E)$$



Probability of an Event Not Occurring (Example)

<u>WEX 13-2-1:</u> Two fair coins are flipped.
Find the probability of not getting two heads.

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Sample Space $S = \{HH, HT, TH, TT\}$

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Sample Space $S = \{HH, HT, TH, TT\}$

Let $E = (\text{Two heads}) = \{HH\}$

Probability of an Event Not Occurring (Example)

WEX 13-2-1: Two fair coins are flipped.
Find the probability of not getting two heads.

Sample Space $S = \{HH, HT, TH, TT\}$

Let $E =$ (Two heads) $= \{HH\}$

Then, $P(\text{Not two heads}) = P(E^c) = 1 - P(E) = 1 - \frac{1}{4} = \boxed{\frac{3}{4}}$

Probability of a Disjunction of Two Events

Proposition

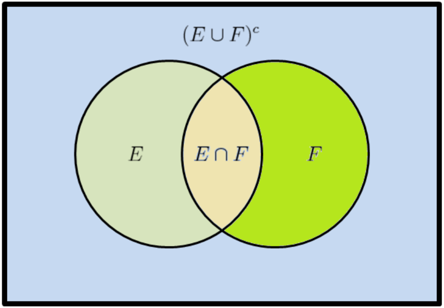
(Probability of a Disjunction of Two Events)

$$P(E \text{ or } F) = P(E) + P(F) - P(E \text{ and } F)$$

which is equivalent to

$$P(E \cup F) = P(E) + P(F) - P(E \cap F)$$

Sample Space (S)



Probability of a Disjunction of Two Events (Example)

WEX 13-2-2: Two fair coins are flipped.
Find the probability for two heads or two tails.

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WEX 13-2-2: Two fair coins are flipped.
Find the probability for two heads or two tails.

Sample Space $S = \{HH, HT, TH, TT\}$

Let $E_1 \equiv$ (Two Heads) $= \{HH\}$
Let $E_2 \equiv$ (Two Tails) $= \{TT\}$

Then, $E_1 \cap E_2 = \{HH\} \cap \{TT\} = \emptyset$

Probability of a Disjunction of Two Events (Example)

WEX 13-2-2: Two fair coins are flipped.
Find the probability for two heads or two tails.

Sample Space $S = \{HH, HT, TH, TT\}$

Let $E_1 \equiv (\text{Two Heads}) = \{HH\}$
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Then, $E_1 \cap E_2 = \{HH\} \cap \{TT\} = \emptyset$

$$P(\text{Two heads or two tails}) = P(E_1 \text{ or } E_2)$$

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$$\begin{aligned} P(\text{Two heads or two tails}) &= P(E_1 \text{ or } E_2) \\ &= P(E_1 \cup E_2) \end{aligned}$$

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$$\begin{aligned} P(\text{Two heads or two tails}) &= P(E_1 \text{ or } E_2) \\ &= P(E_1 \cup E_2) \\ &= P(E_1) + P(E_2) - P(E_1 \cap E_2) \end{aligned}$$

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Mutually Exclusive Events (Definition)

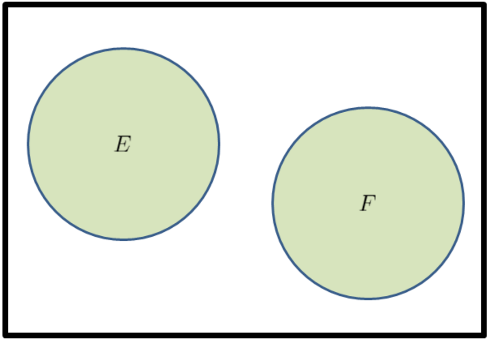
Definition

(Mutually Exclusive Events)

Events E, F are **mutually exclusive** if they have no outcomes in common.

In other words, $E \cap F = \emptyset \iff P(E \cap F) = 0$

Sample Space (S)



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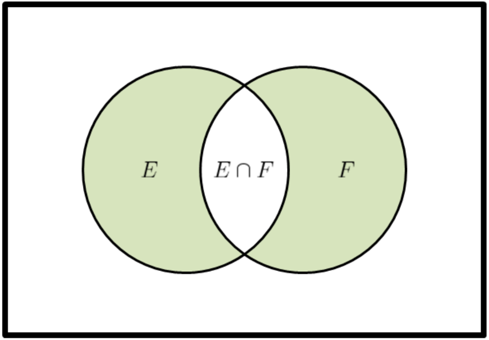


Not Mutually Exclusive

Probability of Two Events Not Occurring

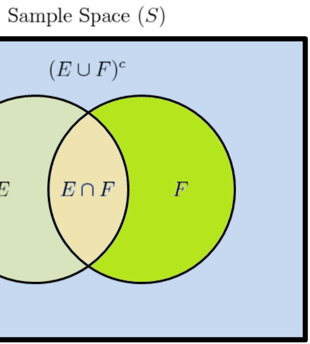
Proposition

(Probability of Two Events Not Occurring)

$$P(\textit{Neither } E \textit{ nor } F) = 1 - P(E) - P(F) + P(E \textit{ and } F)$$

which is equivalent to

$$P\left[(E \cup F)^c\right] = 1 - P(E) - P(F) + P(E \cap F)$$



Probability of Two Events Not Occurring (Example)

WEX 13-2-3: Two fair coins are flipped.
Find the probability that neither the 1^{st} coin is heads nor the 2^{nd} coin is tails.

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Find the probability that neither the 1^{st} coin is heads nor the 2^{nd} coin is tails.

Sample Space $S = \{HH, HT, TH, TT\}$

Let $E_1 \equiv (1^{st}$ coin is heads$) = \{HH, HT\}$
Let $E_2 \equiv (2^{nd}$ coin is tails$) = \{HT, TT\}$

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Let $E_1 \equiv (1^{st}$ coin is heads$) = \{HH, HT\}$
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Then, $E_1 \cap E_2 = \{HH, HT\} \cap \{HT, TT\} = \{HT\}$

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Let $E_2 \equiv (2^{nd}$ coin is tails$) = \{HT, TT\}$

Then, $E_1 \cap E_2 = \{HH, HT\} \cap \{HT, TT\} = \{HT\}$

$$P\begin{pmatrix} \text{Neither } 1^{st} \text{ coin is heads} \\ \text{nor } 2^{nd} \text{ coin is tails} \end{pmatrix} = P(\text{Neither } E_1 \text{ nor } E_2)$$

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WEX 13-2-3: Two fair coins are flipped.
Find the probability that neither the 1^{st} coin is heads nor the 2^{nd} coin is tails.

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Let $E_1 \equiv (1^{st} \text{ coin is heads}) = \{HH, HT\}$
Let $E_2 \equiv (2^{nd} \text{ coin is tails}) = \{HT, TT\}$

Then, $E_1 \cap E_2 = \{HH, HT\} \cap \{HT, TT\} = \{HT\}$

$$\begin{aligned} P\left(\begin{array}{c} \text{Neither } 1^{st} \text{ coin is heads} \\ \text{nor } 2^{nd} \text{ coin is tails} \end{array} \right) &= P(\text{Neither } E_1 \text{ nor } E_2) \\ &= P\left[(E_1 \cup E_2)^c\right] \end{aligned}$$

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Probability of Two Events Not Occurring (Example)

WEX 13-2-3: Two fair coins are flipped.
Find the probability that neither the 1^{st} coin is heads nor the 2^{nd} coin is tails.

Sample Space $S = \{HH, HT, TH, TT\}$

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$$\begin{aligned} P\begin{pmatrix} \text{Neither } 1^{st} \text{ coin is heads} \\ \text{nor } 2^{nd} \text{ coin is tails} \end{pmatrix} &= P(\text{Neither } E_1 \text{ nor } E_2) \\ &= P\left[(E_1 \cup E_2)^c\right] \\ &= 1 - P(E_1) - P(E_2) + P(E_1 \cap E_2) \\ &= 1 - \frac{2}{4} - \frac{2}{4} + \frac{1}{4} \end{aligned}$$

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Let $E_2 \equiv (2^{nd}$ coin is tails$) = \{HT, TT\}$

Then, $E_1 \cap E_2 = \{HH, HT\} \cap \{HT, TT\} = \{HT\}$

$$\begin{aligned}
P\begin{pmatrix} \text{Neither } 1^{st} \text{ coin is heads} \\ \text{nor } 2^{nd} \text{ coin is tails} \end{pmatrix} &= P(\text{Neither } E_1 \text{ nor } E_2) \\
&= P\left[(E_1 \cup E_2)^c\right] \\
&= 1 - P(E_1) - P(E_2) + P(E_1 \cap E_2) \\
&= 1 - \tfrac{2}{4} - \tfrac{2}{4} + \tfrac{1}{4} \\
&= \boxed{\frac{1}{4}}
\end{aligned}$$

One More Example

WEX 13-2-4: Let $P(E \cup F) = 0.30$, $P(E) = 0.15$, $P(F) = 0.25$.

Find: (a) $P(E \cap F)$ (b) $P\left[(E \cup F)^c\right]$ (c) $P\left[(E \cap F)^c\right]$

One More Example

WEX 13-2-4: Let $P(E \cup F) = 0.30$, $P(E) = 0.15$, $P(F) = 0.25$.

Find: (a) $P(E \cap F)$ (b) $P\left[(E \cup F)^c\right]$ (c) $P\left[(E \cap F)^c\right]$

(a) $P(E \cup F) = P(E) + P(F) - P(E \cap F)$

One More Example

WEX 13-2-4: Let $P(E \cup F) = 0.30$, $P(E) = 0.15$, $P(F) = 0.25$.

Find: (a) $P(E \cap F)$ (b) $P\left[(E \cup F)^c\right]$ (c) $P\left[(E \cap F)^c\right]$

(a) $P(E \cup F) = P(E) + P(F) - P(E \cap F)$
$\implies 0.30 = 0.15 + 0.25 - P(E \cap F)$

One More Example

WEX 13-2-4: Let $P(E \cup F) = 0.30$, $P(E) = 0.15$, $P(F) = 0.25$.

Find: (a) $P(E \cap F)$ (b) $P\left[(E \cup F)^c\right]$ (c) $P\left[(E \cap F)^c\right]$

(a) $P(E \cup F) = P(E) + P(F) - P(E \cap F)$
$\implies 0.30 = 0.15 + 0.25 - P(E \cap F)$
$\implies 0.30 = 0.40 - P(E \cap F)$

One More Example

WEX 13-2-4: Let $P(E \cup F) = 0.30$, $P(E) = 0.15$, $P(F) = 0.25$.

Find: (a) $P(E \cap F)$ (b) $P\left[(E \cup F)^c\right]$ (c) $P\left[(E \cap F)^c\right]$

(a) $P(E \cup F) = P(E) + P(F) - P(E \cap F)$
$\implies 0.30 = 0.15 + 0.25 - P(E \cap F)$
$\implies 0.30 = 0.40 - P(E \cap F)$
$\implies -0.10 = -P(E \cap F)$

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(a) $P(E \cup F) = P(E) + P(F) - P(E \cap F)$

$\implies 0.30 = 0.15 + 0.25 - P(E \cap F)$

$\implies 0.30 = 0.40 - P(E \cap F)$

$\implies -0.10 = -P(E \cap F)$

$\implies P(E \cap F) = \boxed{0.10}$

One More Example

WEX 13-2-4: Let $P(E \cup F) = 0.30$, $P(E) = 0.15$, $P(F) = 0.25$.

Find: (a) $P(E \cap F)$ (b) $P\left[(E \cup F)^c\right]$ (c) $P\left[(E \cap F)^c\right]$

(a) $P(E \cup F) = P(E) + P(F) - P(E \cap F)$
$\implies 0.30 = 0.15 + 0.25 - P(E \cap F)$
$\implies 0.30 = 0.40 - P(E \cap F)$
$\implies -0.10 = -P(E \cap F)$
$\implies P(E \cap F) = \boxed{0.10}$

(b) $P\left[(E \cup F)^c\right] = 1 - P(E \cup F)$

One More Example

WEX 13-2-4: Let $P(E \cup F) = 0.30$, $P(E) = 0.15$, $P(F) = 0.25$.

Find: (a) $P(E \cap F)$ (b) $P\left[(E \cup F)^c\right]$ (c) $P\left[(E \cap F)^c\right]$

(a) $P(E \cup F) = P(E) + P(F) - P(E \cap F)$

$\implies 0.30 = 0.15 + 0.25 - P(E \cap F)$

$\implies 0.30 = 0.40 - P(E \cap F)$

$\implies -0.10 = -P(E \cap F)$

$\implies P(E \cap F) = \boxed{0.10}$

(b) $P\left[(E \cup F)^c\right] = 1 - P(E \cup F) = 1 - 0.30 = \boxed{0.70}$

One More Example

WEX 13-2-4: Let $P(E \cup F) = 0.30$, $P(E) = 0.15$, $P(F) = 0.25$.

Find: (a) $P(E \cap F)$ (b) $P\left[(E \cup F)^c\right]$ (c) $P\left[(E \cap F)^c\right]$

(a) $P(E \cup F) = P(E) + P(F) - P(E \cap F)$
$\implies 0.30 = 0.15 + 0.25 - P(E \cap F)$
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$\implies -0.10 = -P(E \cap F)$
$\implies P(E \cap F) = \boxed{0.10}$

(b) $P\left[(E \cup F)^c\right] = 1 - P(E \cup F) = 1 - 0.30 = \boxed{0.70}$

(c) $P\left[(E \cap F)^c\right]$

One More Example

WEX 13-2-4: Let $P(E \cup F) = 0.30$, $P(E) = 0.15$, $P(F) = 0.25$.

Find: (a) $P(E \cap F)$ (b) $P\left[(E \cup F)^c\right]$ (c) $P\left[(E \cap F)^c\right]$

(a) $P(E \cup F) = P(E) + P(F) - P(E \cap F)$

$\implies 0.30 = 0.15 + 0.25 - P(E \cap F)$

$\implies 0.30 = 0.40 - P(E \cap F)$

$\implies -0.10 = -P(E \cap F)$

$\implies P(E \cap F) = \boxed{0.10}$

(b) $P\left[(E \cup F)^c\right] = 1 - P(E \cup F) = 1 - 0.30 = \boxed{0.70}$

(c) $P\left[(E \cap F)^c\right] = 1 - P(E \cap F)$

One More Example

<u>**WEX 13-2-4:**</u> Let $P(E \cup F) = 0.30$, $P(E) = 0.15$, $P(F) = 0.25$.

Find: (a) $P(E \cap F)$ (b) $P\left[(E \cup F)^c\right]$ (c) $P\left[(E \cap F)^c\right]$

(a) $P(E \cup F) = P(E) + P(F) - P(E \cap F)$

$\implies 0.30 = 0.15 + 0.25 - P(E \cap F)$

$\implies 0.30 = 0.40 - P(E \cap F)$

$\implies -0.10 = -P(E \cap F)$

$\implies P(E \cap F) = \boxed{0.10}$

(b) $P\left[(E \cup F)^c\right] = 1 - P(E \cup F) = 1 - 0.30 = \boxed{0.70}$

(c) $P\left[(E \cap F)^c\right] = 1 - P(E \cap F) = 1 - 0.10 = \boxed{0.90}$

Fin.