Conditional Probability, Independence of Events

Contemporary Math

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Conditional Probability (Definition)

The occurrence of one event can affect the probability of another event:

Definition

(Conditional Probability)

Let events E, F be events in a sample space S.

Then:

- The **conditional probability** of F given E, denoted $P(F|E)$, is the probability of event F assuming that event E has already occurred.
- The **conditional probability** of E given F, denoted $P(E|F)$, is the probability of event E assuming that event F has already occurred.

WARNING: **Order matters**: in general, $P(F|E) \neq P(E|F)$

Conditional Probability

But the previous definition is too crude to use.
How does conditional probability relate to ordinary probablity?

Proposition

(Conditional Probability)

Let events E, F be events in a sample space S.

Then:

$$P(\textit{If } E \textit{ then } F) = P(F \textit{ given } E) = P(F|E) = \frac{m(E \cap F)}{m(E)}$$

or equivalently

$$P(\textit{If } E \textit{ then } F) = P(F \textit{ given } E) = P(F|E) = \frac{P(E \cap F)}{P(E)}$$

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Conditional Probability (Example)

<u>**WEX 13-3-1:**</u> A fair coin is flipped and a fair die is rolled.
(a) Find the probability that the coin shows tails given that the die shows 5.
(b) Find the probability that if the coin shows tails then the die shows 5.

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Let event $E \equiv$ (Coin shows tails) $= \{(T,1),(T,2),(T,3),(T,4),(T,5),(T,6)\}$
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(a) $P(E \text{ given } F) = P(E|F) = \dfrac{P(E \cap F)}{P(F)} = \dfrac{1/12}{1/6} = \dfrac{1}{12} \div \dfrac{1}{6} = \dfrac{1}{12} \times \dfrac{6}{1} = \boxed{\dfrac{1}{2}}$

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(b) $P(\text{If } E \text{ then } F) = P(F|E) = \dfrac{P(E \cap F)}{P(E)} = \dfrac{1/12}{1/2} = \dfrac{1}{12} \div \dfrac{1}{2} = \dfrac{1}{12} \times \dfrac{2}{1} = \boxed{\dfrac{1}{6}}$

Intersection of Events (Alternative Formula)

The intersection of two events can found using conditional probability:

Definition

(Intersection of Two Events)

Let events E, F be events in a sample space S.

Then:

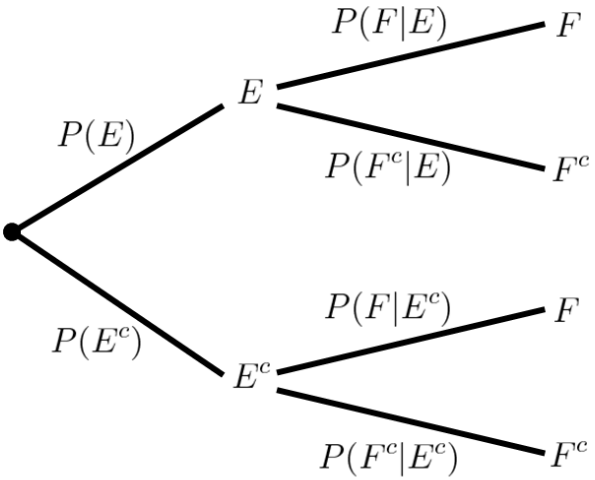
$$P(E \cap F) = P(E) \cdot P(F|E)$$

or equivalently

$$P(E \cap F) = P(F) \cdot P(E|F)$$

Two-Stage Experiments & Probability Trees

Some experiments are conducted in **two stages**.
Such two-stage experiments can be visualized using a **probability tree**:



Independence of Events (Definition)

Often it's important to know if two events depend on each other or not:

Definition

(Independent Events)

Let events E, F be events in a sample space S.

Then, events E and F are **independent** if:

$$P(F|E) = P(F)$$

or equivalently

$$P(E|F) = P(E)$$

or equivalently

$$P(E \cap F) = P(E) \cdot P(F)$$

Otherwise, events E and F are **dependent**.

Independence of Events (Example)

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Are the events ”Coin shows tails” and ”Die shows 5” independent?

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Let event $E \equiv$ (Coin shows tails) $= \{(T,1),(T,2),(T,3),(T,4),(T,5),(T,6)\}$
Let event $F \equiv$ (Die shows 5) $= \{(H,5),(T,5)\}$

Then $E \cap F = \{(T,5)\}$

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To check independence of the two events,
it's easiest to check the relation $P(E \cap F) = P(E) \cdot P(F)$:

$$P(E) \cdot P(F) = \frac{1}{2} \times \frac{1}{6} = \frac{1}{12} = P(E \cap F)$$

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Therefore, since $P(E \cap F) = P(E) \cdot P(F)$, events E, F are $\boxed{\text{independent}}$

Independence of Events (Example)

WEX 13-3-3: A fair coin is flipped and a fair die is rolled.
Are the events ”Die shows 3” and ”Die shows 5” independent?

Independence of Events (Example)

<u>**WEX 13-3-3:**</u> A fair coin is flipped and a fair die is rolled.
Are the events "Die shows 3" and "Die shows 5" independent?

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Let event $E \equiv$ (Die shows 3) $= \{(H,3),(T,3)\}$
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Independence of Events (Example)

WEX 13-3-3: A fair coin is flipped and a fair die is rolled.
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Let event $E \equiv$ (Die shows 3) $= \{(H,3),(T,3)\}$
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Then $E \cap F = \emptyset$

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Therefore, since $P(E \cap F) \neq P(E) \cdot P(F)$, events E, F are $\boxed{\text{dependent}}$

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