Graph Theory: Euler Paths, Euler Circuits

Contemporary Math

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TTU

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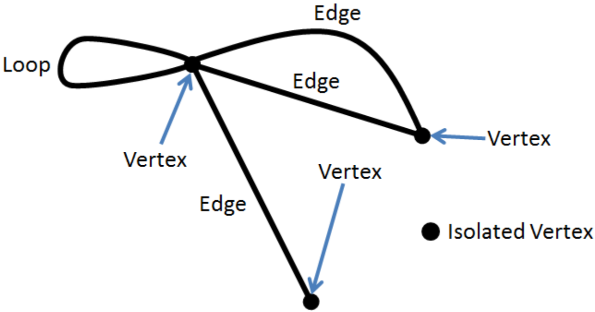
Graphs, Vertices, Edges (Definition)

Definition

(Graph, Vertex, Edge)

A **graph** consists of a finite set of points, called **vertices**, and lines/curves, called **edges**, that join pairs of vertices.

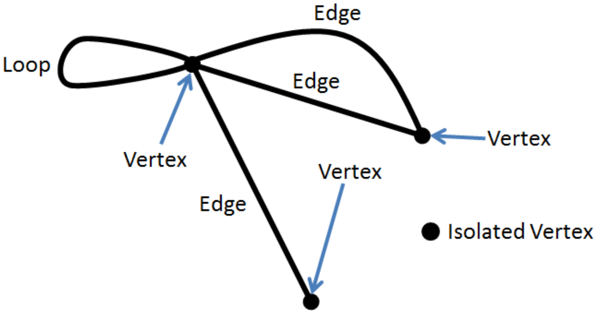


Isolated Vertices, Loops (Definition)

Definition

(Isolated Vertex, Loop)

An **isolated vertex** has no edges joined to it.
A **loop** is an edge which joins one vertex with itself.



Connected Graphs & Bridges (Definition)

Definition

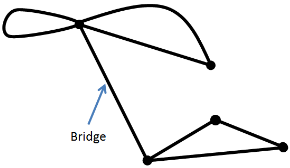
(Connected Graph)

A graph is **connected** if it's possible to travel from any vertex to any other vertex of the graph by moving along successive edges.

Definition

(Bridge)

An edge of a connected graph is a **bridge** if removing the edge causes the graph to no longer be connected.



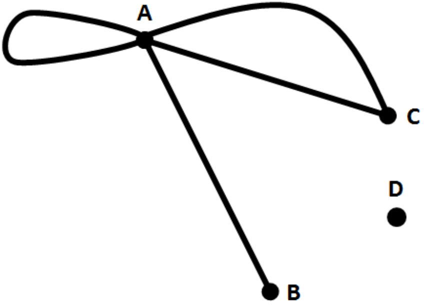
Connected Graph

Degree of a Vertex (Definition)

Definition

(Degree of a Vertex)

The **degree** of a vertex is the # of edges joined to that vertex.
Loops count as **two edges**.
The degree of an **isolated vertex** is defined to be **zero**.



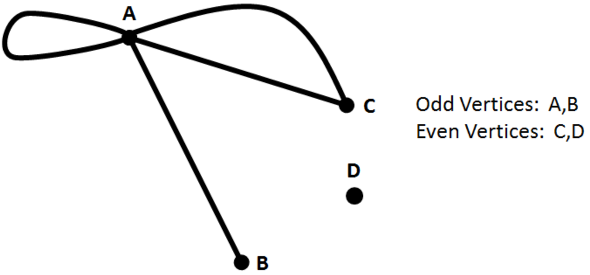
deg(A) = 5
deg(B) = 1
deg(C) = 2
deg(D) = 0

Odd & Even Vertices (Definition)

Definition

(Odd Vertex, Even Vertex)

An **odd vertex** is a vertex with an odd degree.
An **even vertex** is a vertex with an even degree.



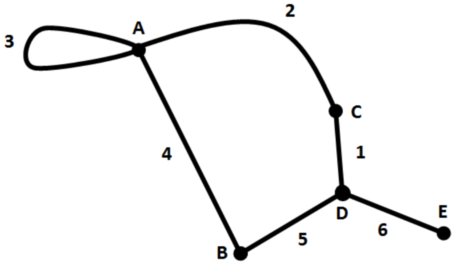
Euler Paths & Euler Circuits (Definition)

Definition

(Path, Euler Path, Euler Circuit)

A **path** is a sequence of consecutive edges in which no edge is repeated.
The **length** of a path is the # of edges in the path.

An **Euler path** is a path that contains all edges of the graph.
An **Euler circuit** is an Euler path that begins & ends at the same vertex.



Euler Path from D to E: DCAABDE

Euler Paths & Euler Circuits (Definition)

Definition

(Path, Euler Path, Euler Circuit)

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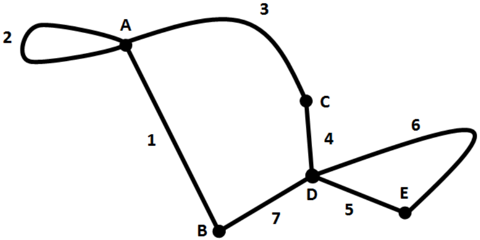


Euler Circuit starting at B: BAACDEDB

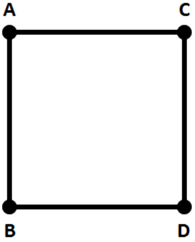
Tracing a Graph (Definition)

Definition

(Tracing a Graph)

To **trace** a graph means to begin at a vertex & draw the entire graph provided:
(1) The pencil is never lifted off the paper
(2) Each edge is not traversed more than once

A graph that can be traced as described above is called **traceable.**



Traceable Graph
(starting from any vertex)

Tracing a Graph (Definition)

Definition

(Tracing a Graph)

To **trace** a graph means to begin at a vertex & draw the entire graph provided:
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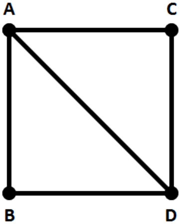

Traceable Graph
(starting from A or D)

Tracing a Graph (Definition)

Definition

(Tracing a Graph)

To **trace** a graph means to begin at a vertex & draw the entire graph provided:
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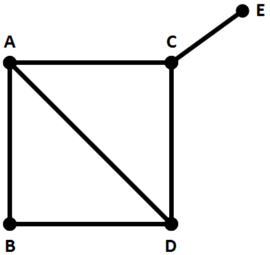
(2) Each edge is not traversed more than once

A graph that can be traced as described above is called **traceable**.



Non-Traceable Graph

Existence of Euler Path/Circuit (Theorem)

Theorem

(Euler's Theorem)

A graph can be traced if it's connected and has zero or two odd vertices.

Corollary

(Corollary to Euler's Theorem)

A graph is traceable if it contains an Euler path or Euler circuit.

Corollary

(Corollary to Euler's Theorem)

(a) *Suppose a connected graph has **two odd vertices**.*
Then, the tracing must begin at one odd vertex and end at the other.
*i.e. The tracing is an **Euler path** from one odd vertex to the other.*

(b) *Suppose a connected graph has **zero odd vertices**.*
Then, the tracing must begin and end at the same vertex.
*i.e. The tracing is an **Euler circuit**.*

Finding an Euler Circuit (Fleury's Algorithm)

Very often, an Euler circuit can be found painlessly by trial-and-error.
However, for ”very large” graphs, trial-and-error is not feasible.
Fleury's Algorithm will systematically find an Euler circuit:

Proposition

(Fluery's Algorithm for finding an Euler Circuit)

INPUT: A connected graph with zero odd vertices. (i.e. all vertices are even)

(1) Start at any vertex.
(2) Traverse an edge, but not a bridge unless there's no alternative.
Mark/erase/cover-up the traversed edge.
(3) Repeat (2) until there's no more edges to traverse.

OUTPUT: An Euler circuit.

Finding an Euler Path (Fleury's Algorithm)

Very often, an Euler path can be found painlessly by trial-and-error.
However, for ”very large” graphs, trial-and-error is not feasible.
Fleury's Algorithm will systematically find an Euler path:

Proposition

(Fluery's Algorithm for finding an Euler Path)

INPUT: *A connected graph with two odd vertices.*

(1) Start at one of the two odd vertices.
(2) Traverse an edge, but not a bridge unless there's no alternative.
Mark/erase/cover-up the traversed edge.
(3) Repeat (2) until there's no more edges to traverse.

OUTPUT: *An Euler path from one odd vertex to the other odd vertex.*

Fin

Fin.