Interest

Contemporary Math

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TTU

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Loans & Interest (Definitions)

Suppose you wish to buy a very expensive product or service (e.g. house, car, college education, ...), but you don't have enough money to comfortably buy it.

What should you do if you absolutely must have the product/service now??

BORROW MONEY!

But what incentive does a person* or institution have to lend you the money??

THEY EARN INTEREST OVER TIME VIA A LOAN!

Definition

Interest is the money that a **borrower** pays to a **lender** to use the lender's money, called the **principal**.
A **loan** is the process in which the borrower pays the principal back to the lender plus any accrued interest after a fixed period of time (**loan period**.)
Loans are typically embodied in a legal document called a **promissory note**.

*other than a family member.

Interest-Bearing Loans/Accounts*

Loan	**Lender**	**Borrower**	**Annual Interest Rate****
Savings Account	Consumer	Bank	0.07%
Money Market Acct	Consumer	Bank	0.40%
CD	Consumer	Bank	4%
Municipal Bond	Consumer	City	8%
Home Mortgage	Bank	Consumer	4%
Car Loan	Bank	Consumer	6.5%
Student Loan	Government	Student	7%
Dealer Loan	Dealer	Consumer	12%
Credit Card Account	Credit Issuer	Consumer	23%
Bail	Bondsman	Suspect	50%
Pawnshop Loan	Pawnshop	Consumer	150%
Payday Loan	Payday Store	Consumer	300%
1940's Mob Loan	Loan Shark	Consumer	1000%

* Not interest-bearing: Stocks, Taxes, Inflation.

** Rates shown may not necessarily reflect current rates.

Interest Rates (Lexicon)

Very often, interest rates are **annual**, meaning the rate is applied once a year, so the forthcoming formulas will assume an **annual interest rate**.

Sometimes the given interest rate is not annual – so how to convert??

Type of Interest Rate	How to Convert to Annual Interest Rate
Semi-Annual	Multiply by 2
Quarterly	Multiply by 4
Monthly	Multiply by 12
Weekly	Multiply by 52
Daily	Multiply by 365

Examples:

- 3% quarterly = 12% annually
- 10% weekly = 520% annually
- 0.2% daily = 73% annually

So, when comparing interest rates, first convert them to annual rates.

Loan Periods (Lexicon)

Formulas involving interest require that the loan period be in **years**.

Sometimes loan periods are in other time units – so how to convert??

Time Unit	How to Convert to Years
Quarter	Divide by 4
Month	Divide by 12
Week	Divide by 52
Day	Divide by 365

Examples:

- 5 quarters = $\frac{5}{4}$ years
- 7 months = $\frac{7}{12}$ year
- 24 months = $\frac{24}{12}$ = 2 years
- 10 weeks = $\frac{10}{52}$ year
- 100 days = $\frac{100}{365}$ year

So, when using interest formulas, first convert loan period to years.

Simple Interest (Definition)

So how to determine the amount of interest?

Here's the simplest way (no pun intended):

Definition

(**Simple Interest** Earned) $= (Principal) \times (Interest\ Rate) \times (Time)$

Proposition

(Simple Interest Formula)

$$I = Prt$$

where
$I \equiv$ *(Simple) Interest earned*
$P \equiv$ *Principal*
$r \equiv$ *Annual Interest Rate*
$t \equiv$ *Loan Period (**in years**)*

The symbol $\equiv$ means "represents".

Future Value using Simple Interest

Definition

(**Future Value** of Account) $= (Principal) + (Interest\ Earned)$

Proposition

(Future Value using Simple Interest)

$$FV = P(1 + rt)$$

where
$FV \equiv$ ***Future Value*** *of Account*
$P \equiv$ *Principal (AKA* ***Present Value*** *of Account)*
$r \equiv$ *Annual Interest Rate*
$t \equiv$ *Loan Period (**in years**)*

The symbol $\equiv$ means "represents".

Simple Interest (Example)

WEX 8-2-1:

A $1000 loan collects simple interest for 5 years at an annual rate of 8%.

(a) How much money is paid back to the lender after 5 years?

(b) How much interest is accrued after 5 years?

Simple Interest (Example)

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$$P = \$1000,\ r = 8\% = 0.08,\ t = 5 \text{ yrs}$$

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$$P = \$1000,\ \ r = 8\% = 0.08,\ \ t = 5 \text{ yrs}$$

(a) $FV = P(1 + rt) = 1000[1 + (0.08)(5)] = 1000[1 + 0.4] = 1000(1.4) = \boxed{\$1400}$

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(b) $I = Prt = 1000(0.08)(5) = 1000(0.4) = \boxed{\$400}$

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SANITY CHECK: $(Future\ Value) - (Simple\ Interest) = (Principal)$

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(b) $I = Prt = 1000(0.08)(5) = 1000(0.4) = \boxed{\$400}$

SANITY CHECK: $(\mathit{Future\ Value}) - (\mathit{Simple\ Interest}) = (\mathit{Principal})$

$\implies \quad FV - I = \$1400 - \$400 = \$1000 = P \quad \checkmark$

Simple Interest (Example)

WEX 8-2-2:
A borrower pays back \$8025 at the end of a 2-yr loan with 9% annual interest. Assuming simple interest, how much money was borrowed?

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$$FV = \$8025,\ \ r = 9\% = 0.09,\ \ t = 2 \text{ yrs}$$

Simple Interest (Example)

WEX 8-2-2:

A borrower pays back \$8025 at the end of a 2-yr loan with 9% annual interest. Assuming simple interest, how much money was borrowed?

$$FV = \$8025, \; r = 9\% = 0.09, \; t = 2 \text{ yrs}$$

$$\begin{aligned}
FV &= P(1+rt) && \leftarrow \text{(Identify relevant formula)} \\
8025 &= P[1+(0.09)(2)] && \leftarrow \text{(Plug in all known quantities)} \\
8025 &= P(1+0.18) && \leftarrow \text{(Simplify)} \\
8025 &= P(1.18) && \leftarrow \text{(Simplify)} \\
\tfrac{8025}{1.18} &= P && \leftarrow \text{(Solve for } P\text{)} \\
6800.847458 &= P && \leftarrow \text{(Use Calculator for tedious arithmetic)} \\
\$6800.85 &= P && \leftarrow \text{(Round to the nearest penny)}
\end{aligned}$$

$$\therefore \quad P = \boxed{\$6800.85}$$

Compound Interest

Here's a more common scheme for computing interest:

Proposition

(Future Value using Compound Interest)

$$FV = P\left(1 + \frac{r}{m}\right)^n$$

$FV \equiv$ ***Future Value*** *of Account*
$P \equiv$ *Principal (AKA* ***Present Value*** *of Account)*
$r \equiv$ *Annual Interest Rate*
$t \equiv$ *Loan Period (**in years**)*
$m \equiv$ *Frequency of Compounding per Year*
$n \equiv$ *Total Number of Payments* $\quad (n = mt)$

The symbol $\equiv$ means "represents".

Frequency of compounding per year (Lexicon)

Compounded annually	$m = 1$
Compounded semi-annually	$m = 2$
Compounded quarterly	$m = 4$
Compounded bimonthly	$m = 6$
Compounded monthly	$m = 12$
Compounded semi-monthly	$m = 24$
Compounded weekly	$m = 52$
Compounded daily	$m = 365$
Compounded 7 times a year	$m = 7$
Compounded 15 times a year	$m = 15$
Compounded 137 times a year	$m = 137$

Compound Interest (Example)

WEX 8-2-3:

A \$1000 loan collects interest for 5 years at an annual rate of 8%, **compounded annually**.

(a) How much money is paid back to the lender after 5 years?

(b) How much interest is accrued after 5 years?

Compound Interest (Example)

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A \$1000 loan collects interest for 5 years at an annual rate of 8%, **compounded annually**.

(a) How much money is paid back to the lender after 5 years?
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$$P = \$1000,\ r = 8\% = 0.08,\ t = 5 \text{ yrs},\ m = 1,\ n = mt = (1)(5) = 5$$

Compound Interest (Example)

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A \$1000 loan collects interest for 5 years at an annual rate of 8%, **compounded annually**.

(a) How much money is paid back to the lender after 5 years?

(b) How much interest is accrued after 5 years?

$$P = \$1000,\ r = 8\% = 0.08,\ t = 5 \text{ yrs},\ m = 1,\ n = mt = (1)(5) = 5$$

(a) $FV = P\left(1 + \frac{r}{m}\right)^n = 1000\left(1 + \frac{0.08}{1}\right)^{(1)(5)} = 1000(1.08)^5 =$
$1000(1.469328077) = 1469.328077 \approx \boxed{\$1469.33}$

(b) $I = FV - P = 1469.33 - 1000 = \boxed{\$469.33}$

Compound Interest (Example)

WEX 8-2-4:

A \$1000 loan collects interest for 5 years at an annual rate of 8%, **compounded weekly**.

(a) How much money is paid back to the lender after 5 years?

(b) How much interest is accrued after 5 years?

Compound Interest (Example)

WEX 8-2-4:
A \$1000 loan collects interest for 5 years at an annual rate of 8%, **compounded weekly**.

(a) How much money is paid back to the lender after 5 years?
(b) How much interest is accrued after 5 years?

$$P = \$1000,\ r = 8\% = 0.08,\ t = 5 \text{ yrs},\ m = 52,\ n = mt = (52)(5) = 260$$

Compound Interest (Example)

WEX 8-2-4:

A \$1000 loan collects interest for 5 years at an annual rate of 8%, **compounded weekly**.

(a) How much money is paid back to the lender after 5 years?

(b) How much interest is accrued after 5 years?

$$P = \$1000,\ r = 8\% = 0.08,\ t = 5 \text{ yrs},\ m = 52,\ n = mt = (52)(5) = 260$$

(a) $FV = P\left(1+\frac{r}{m}\right)^n = 1000\left(1+\frac{0.08}{52}\right)^{(52)(5)} = 1000(1.001538462)^{260} =$
$1000(1.491366215) = 1491.366215 \approx \boxed{\$1491.37}$

(b) $I = FV - P = 1491.37 - 1000 = \boxed{\$491.37}$

Comparing Simple Interest & Compound Interest

$$P = \$1000, \;\; r = 8\%, \;\; t = 5 \text{ yrs}$$

Simple Interest:	$\$400.00$
Compound Interest (annually):	$\$469.33$
Compound Interest (weekly):	$\$491.37$

Comparing Simple Interest & Compound Interest

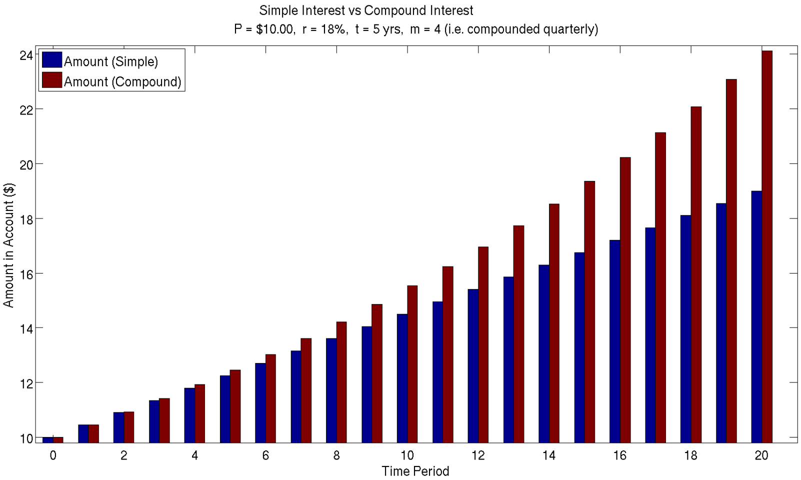
$n \equiv$ (# of Compounding Periods) $\quad n = mt \quad$ "Pd" $\equiv$ "Period"

Time	Simple Interest	Compound Interest
Deposit	$S_0 = P$	$C_0 = P$
1^{st} Pd	$S_1 = S_0 + \left(\frac{r}{m}\right) P = P\left(1 + \frac{r}{m}\right)$	$C_1 = C_0 + \left(\frac{r}{m}\right) C_0 = P\left(1 + \frac{r}{m}\right)$
2^{nd} Pd	$S_2 = S_1 + \left(\frac{r}{m}\right) P = P\left(1 + \frac{2r}{m}\right)$	$C_2 = C_1 + \left(\frac{r}{m}\right) C_1 = P\left(1 + \frac{r}{m}\right)^2$
3^{rd} Pd	$S_3 = S_2 + \left(\frac{r}{m}\right) P = P\left(1 + \frac{3r}{m}\right)$	$C_3 = C_2 + \left(\frac{r}{m}\right) C_2 = P\left(1 + \frac{r}{m}\right)^3$
4^{th} Pd	$S_4 = S_3 + \left(\frac{r}{m}\right) P = P\left(1 + \frac{4r}{m}\right)$	$C_4 = C_3 + \left(\frac{r}{m}\right) C_3 = P\left(1 + \frac{r}{m}\right)^4$
$\vdots$	$\vdots$	$\vdots$
n^{th} Pd	$S_n = S_{n-1} + \left(\frac{r}{m}\right) P = P\left(1 + \frac{nr}{m}\right)$	$C_n = C_{n-1} + \left(\frac{r}{m}\right) C_{n-1} = P\left(1 + \frac{r}{m}\right)^n$

Using Simple Interest: $FV = S_n = P\left(1 + \frac{nr}{m}\right) = P\left(1 + \frac{(mt)r}{m}\right) = P(1 + rt)$

Using Compound Interest: $FV = C_n = P\left(1 + \frac{r}{m}\right)^n = P\left(1 + \frac{r}{m}\right)^{mt}$

Comparing Simple Interest & Compound Interest



All else being equal, compound interest grows faster than simple interest.

Simple interest is often used only for **short-term notes**. (1 year or less)

Exponents & Roots (Review)

Recall from Algebra the properties of **exponents** & **roots**:

Proposition

(Properties of Exponents)

$$a^{m+n} = a^m a^n \qquad a^{m-n} = \frac{a^m}{a^n} \qquad (a^m)^n = a^{mn} \qquad a^{-m} = \frac{1}{a^m}$$

$$(ab)^m = a^m b^m \qquad \left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$$

Proposition

(Properties of Roots) n, m are ***positive integers***

$$\sqrt[n]{x} = x^{1/n} \qquad \sqrt[n]{xy} = \sqrt[n]{x}\sqrt[n]{y} \qquad \sqrt[n]{\frac{x}{y}} = \frac{\sqrt[n]{x}}{\sqrt[n]{y}} \qquad \sqrt[m]{\sqrt[n]{x}} = \sqrt[mn]{x}$$

<u>WARNINGS:</u>

- Even roots of negative #'s are **undefined**: $\sqrt{-2}$, $\sqrt[4]{-9}$, $\sqrt[8]{-1}$, ...
- In general, $\sqrt[n]{x+y} \neq \sqrt[n]{x} + \sqrt[n]{y}$ (e.g. $\sqrt{2} \approx 1.414$ yet $\sqrt{1} + \sqrt{1} = 2$)

Solving Equations of the Form $x^a = b$ for x

$$FV = P\left(1+\frac{r}{m}\right)^n$$

Since **compound interest** involves an **power** (n), solving compound interest problems for the **interest rate** (r) involves solving eqn's of the form $x^a = b$:

Proposition

(Solving a Power Equation)

Given the ***power equation***

$$x^a = b$$

Then:

$(x^a)^{1/a} = (b)^{1/a}$	*Raise both sides to the* ***reciprocal power*** $1/a$
$x^{(a)(1/a)} = b^{1/a}$	*Property of Exponents*
$x = b^{1/a}$	*Simplify*

Compound Interest (Example)

<u>**WEX 8-2-5:**</u>
A $1000 loan collects interest for 20 months, **compounded monthly**.
At the end of the 20 months, $1400 is payed back to the lender.
What is the annual interest rate?

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$$P = \$1000,\ FV = \$1400,\ t = \frac{20}{12} \text{ yrs},\ m = 12,\ n = mt = (12)\left(\frac{20}{12}\right) = 20$$

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At the end of the 20 months, \$1400 is payed back to the lender.
What is the annual interest rate?

$$P = \$1000,\ FV = \$1400,\ t = \tfrac{20}{12} \text{ yrs},\ m = 12,\ n = mt = (12)\left(\tfrac{20}{12}\right) = 20$$

$$\begin{aligned}
FV &= P\left(1+\frac{r}{m}\right)^n && \leftarrow \text{(Identify relevant formula)}\\
1400 &= 1000\left(1+\frac{r}{12}\right)^{20} && \leftarrow \text{(Plug in all known quantities)}\\
\frac{1400}{1000} &= \left(1+\frac{r}{12}\right)^{20} && \leftarrow \text{(Divide both sides by Principal)}\\
1.4 &= \left(1+\frac{r}{12}\right)^{20} && \leftarrow \text{(Simplify LHS)}\\
1.4^{1/20} &= \left[\left(1+\frac{r}{12}\right)^{20}\right]^{1/20} && \leftarrow \text{(Take reciprocal power)}\\
1.016965926 &= 1+\frac{r}{12} && \leftarrow \text{(Compute power (use calculator))}
\end{aligned}$$

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$$\begin{aligned}
1.4^{1/20} &= \left[\left(1+\frac{r}{12}\right)^{20}\right]^{1/20} && \leftarrow \text{(Take reciprocal power)}\\
1.016965926 &= 1+\frac{r}{12} && \leftarrow \text{(Compute power (use calculator))}\\
0.016965926 &= \frac{r}{12} && \leftarrow \text{(Subtract both sides by 1)}\\
0.203591109 &= r && \leftarrow \text{(Multiply both sides by 12)}\\
0.2040 &= r && \leftarrow \text{(Round to four decimal places)}\\
20.40\% &= r && \leftarrow \text{(Convert decimal to percent)}
\end{aligned}$$

$$\therefore\quad r = \boxed{20.40\%}$$

Fin.