

How to painlessly solve SAS and SSS triangles

The textbook [1] treats the SAS and SSS layouts as follows:

- * SAS layout : First, use Law of Cosines to find side opposite of known angle.
Then, use Law of Sines to find the smaller of the two unknown angles.
Finally, use the Angle Sum of Triangles to find the last angle.
- * SSS layout : First, use Law of Cosines to find the largest angle.
Then, find either remaining angle using the Law of Sines.
Finally, use the Angle Sum to find the last angle.

Notice that these two methods use both the Law of Cosines and the Law of Sines.

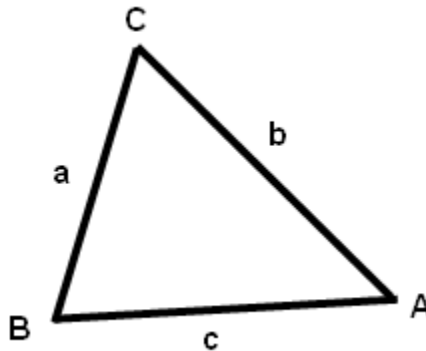
If these methods are not followed in the correct sequence, ambiguity (similar to the SSA case) can occur, potentially causing wrong answers or at least more time and work.

It turns out if one strictly uses Law of Cosines throughout in either case, no ambiguities will ever occur.

REMARK: It's possible that for a SSS layout, the triangle may not actually exist! See the 3rd example below.

What follows are three examples of exclusively using Law of Cosines to solve SAS & SSS triangles while preventing ambiguity.

Triangle Labeling Convention used by the textbook [1]:



Angle Sum of Triangles: $A + B + C = 180^\circ$

Law of Sines: $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$

Law of Cosines: $a^2 = b^2 + c^2 - 2bc \cos A$
 $b^2 = a^2 + c^2 - 2ac \cos B$
 $c^2 = a^2 + b^2 - 2ab \cos C$

Triangle Inequality (pg 319): $a + b > c$
 $b + c > a$ (i.e. The sum of any two sides must be greater than the remaining side)
 $c + a > b$

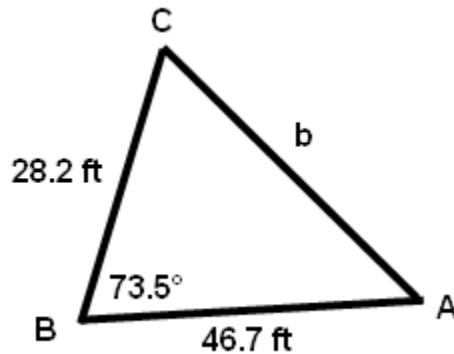
Heron's Area Formula (pg 323): $\text{Area} = \sqrt{s(s-a)(s-b)(s-c)}$, where $s = \frac{1}{2}(a + b + c)$

Example (E8): Solve triangle ABC with $B = 73.5^\circ$, $a = 28.2$ ft, $c = 46.7$ ft

STEP 0: MAKE SURE CALCULATOR IS SET TO **DEGREE** MODE!

STEP 1: Sketch triangle ABC (not to scale) and determine layout

Upon sketching triangle ABC and labeling what is given using the standard convention in the textbook [1], one realizes that this is an **SAS** case, implying that the **Law of Cosines** must be invoked.



STEP 2: Find side b

$$\begin{aligned}b^2 &= a^2 + c^2 - 2ac \cos B \\ \Rightarrow b^2 &= (28.2 \text{ ft})^2 + (46.7)^2 - 2(28.2 \text{ ft})(46.7 \text{ ft}) \cos 73.5^\circ \\ \Rightarrow b^2 &= 2228.067664 \text{ ft}^2 \\ \Rightarrow b &= 47.20241163 \text{ ft} \\ \Rightarrow b &= \mathbf{47.2 \text{ ft}} \text{ (significant figures)}\end{aligned}$$

STEP 3: Find one of the angles, say A

$$\begin{aligned}a^2 &= b^2 + c^2 - 2bc \cos A \\ \Rightarrow (28.2 \text{ ft})^2 &= (47.20241163 \text{ ft})^2 + (46.7 \text{ ft})^2 - 2(47.20241163 \text{ ft})(46.7 \text{ ft}) \cos A \\ \Rightarrow \cos A &= 0.819677758 \\ \Rightarrow A &= 34.94745091^\circ \\ \Rightarrow A &= \mathbf{34.9^\circ} \text{ (significant figures)}\end{aligned}$$

STEP 4: Find the other angle (in this case, C)

The easiest method is to use the Angle Sum of Triangles:

$$A + B + C = 180^\circ \Rightarrow 34.9^\circ + 73.5^\circ + C = 180^\circ \Rightarrow C = \mathbf{71.6^\circ}$$

or if consistency is paramount to you, then apply Law of Cosines a third time:

$$\begin{aligned}c^2 &= a^2 + b^2 - 2ab \cos C \\ \Rightarrow (46.7 \text{ ft})^2 &= (28.2 \text{ ft})^2 + (47.20241163 \text{ ft})^2 - 2(28.2 \text{ ft})(47.20241163 \text{ ft}) \cos C \\ \Rightarrow \cos C &= 0.316434752 \\ \Rightarrow C &= 71.55254979^\circ \\ \Rightarrow C &= \mathbf{71.6^\circ} \text{ (significant figures)}\end{aligned}$$

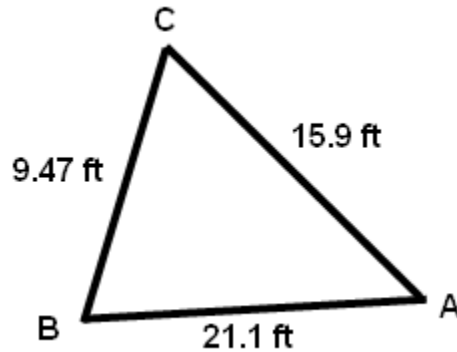
REMARK: Notice that when using intermediate calculations, the entire value is used, not the rounded form!! This ensures that round-off error will not occur once reaching the final answer.

Example (E9): Solve triangle ABC with $a = 9.47$ ft, $b = 15.9$ ft, $c = 21.1$ ft

STEP 0: MAKE SURE CALCULATOR IS SET TO **DEGREE** MODE!

STEP 1: Sketch triangle ABC (not to scale) and determine layout

Upon sketching triangle ABC and labeling what is given using the standard convention in the textbook [1], one realizes that this is an SSS case, implying that the **Law of Cosines** must be invoked.



STEP 2: Find one angle (doesn't matter which one is chosen):

$$\begin{aligned} b^2 &= a^2 + c^2 - 2ac \cos B \\ \Rightarrow (15.9 \text{ ft})^2 &= (9.47 \text{ ft})^2 + (21.1 \text{ ft})^2 - 2(9.47 \text{ ft})(21.1 \text{ ft})\cos B \\ \Rightarrow \cos B &= 0.705848101 \\ \Rightarrow \mathbf{B} &= \mathbf{45.1^\circ} \text{ (significant figures)} \end{aligned}$$

STEP 3: Find another angle (again, doesn't matter which one is chosen):

$$\begin{aligned} a^2 &= b^2 + c^2 - 2bc \cos A \\ \Rightarrow (9.47 \text{ ft})^2 &= (15.9 \text{ ft})^2 + (21.1 \text{ ft})^2 - 2(15.9 \text{ ft})(21.1 \text{ ft})\cos A \\ \Rightarrow \cos A &= 0.906642672 \\ \Rightarrow A &= 24.95455988^\circ \\ \Rightarrow \mathbf{A} &= \mathbf{25.0^\circ} \text{ (significant figures)} \end{aligned}$$

STEP 4: Find the last angle:

The easiest method is to use the Angle Sum of Triangles:

$$A + B + C = 180^\circ \Rightarrow 25.0^\circ + 45.1^\circ + C = 180^\circ \Rightarrow \mathbf{C = 109.9^\circ}$$

or if consistency is paramount to you, then apply Law of Cosines a third time:

$$\begin{aligned} c^2 &= a^2 + b^2 - 2ab \cos C \\ \Rightarrow (21.1 \text{ ft})^2 &= (9.47 \text{ ft})^2 + (15.9 \text{ ft})^2 - 2(9.47 \text{ ft})(15.9 \text{ ft})\cos C \\ \Rightarrow \cos C &= -0.3410940208 \\ \Rightarrow C &= 109.9435417^\circ \\ \Rightarrow \mathbf{C} &= \mathbf{109.9^\circ} \text{ (1 decimal place so that } A + B + C = 180^\circ) \end{aligned}$$

REMARK: Notice that when using intermediate calculations, the entire value is used, not the rounded form!! This ensures that round-off error will not occur once reaching the final answer.

Example (E10): Solve triangle ABC with $a = 25$ yd, $b = 12$ yd, $c = 10$ yd

STEP 0: MAKE SURE CALCULATOR IS SET TO **DEGREE** MODE!

STEP 1: Sketch triangle ABC (not to scale) and determine layout

Upon sketching triangle ABC and labeling what is given using the standard convention in the textbook [1], one realizes that this is an **SSS** case, implying that the **Law of Cosines** must be invoked.

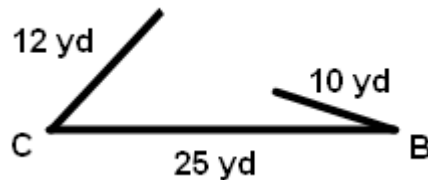
STEP 2: Find one angle (doesn't matter which one is chosen):

$$\begin{aligned} b^2 &= a^2 + c^2 - 2ac \cos B \\ \Rightarrow (12 \text{ yd})^2 &= (25 \text{ yd})^2 + (10 \text{ yd})^2 - 2(25 \text{ yd})(10 \text{ yd}) \cos B \\ \Rightarrow \cos B &= 1.162 \\ \Rightarrow \text{DOMAIN ERROR} &(\text{message on calculator}) \end{aligned}$$

Since 1.162 falls outside the range of cosine $[-1, 1]$, NO TRIANGLE EXISTS -- DONE!

No triangle exists?? What happened???

If this triangle is drawn very carefully to scale, the picture looks something like this:



Sides b and c are too short to connect and form angle A -- regardless of the choices for angles B and C.

The triangle fails the Triangle Inequality since $b + c < a \Leftrightarrow 12 \text{ yd} + 10 \text{ yd} < 25 \text{ yd} \Leftrightarrow 22 \text{ yd} < 25 \text{ yd}$
(NOTE: It suffices to show one of the three inequalities fails to hold in order to show that a triangle does not exist.)

Hence, "triangle ABC" is really a 'false triangle' -- in other words, triangle ABC does not truly exist!

In case you're wondering, using the Law of Cosines to find the other two angles yield the same disastrous result:

$$\begin{aligned} a^2 &= b^2 + c^2 - 2bc \cos A \\ \Rightarrow (25 \text{ yd})^2 &= (12 \text{ yd})^2 + (10 \text{ yd})^2 - 2(12 \text{ yd})(10 \text{ yd}) \cos A \\ \Rightarrow \cos A &= -1.5875 \\ \Rightarrow \text{DOMAIN ERROR} &(\text{message on calculator}) \end{aligned}$$

$$\begin{aligned} c^2 &= a^2 + b^2 - 2ab \cos C \\ \Rightarrow (10 \text{ yd})^2 &= (25 \text{ yd})^2 + (12 \text{ yd})^2 - 2(25 \text{ yd})(12 \text{ yd}) \cos C \\ \Rightarrow \cos C &= 1.115 \\ \Rightarrow \text{DOMAIN ERROR} &(\text{message on calculator}) \end{aligned}$$

Finally, an attempt to find the area of this 'false triangle' via Heron's Formula (pg 323) is futile:

$$\begin{aligned} \text{Semiperimeter } s &= \frac{1}{2}(a + b + c) = \frac{1}{2}(25 \text{ yd} + 12 \text{ yd} + 10 \text{ yd}) = 23.5 \text{ yd} \\ \text{Area} &= \sqrt{s(s-a)(s-b)(s-c)} = \sqrt{(23.5 \text{ yd})(23.5 \text{ yd} - 25 \text{ yd})(23.5 \text{ yd} - 12 \text{ yd})(23.5 \text{ yd} - 10 \text{ yd})} \\ &= \sqrt{-5472.5625 \text{ yd}^4} = \text{UNDEFINED} \text{ (Since radicand is negative)} \end{aligned}$$

References

- [1] M. L. Lial, J. E. Hornsby, D. Schneider. *Trigonometry*. Pearson, Boston, MA, 9th Edition, 2009.