Texas Tech University
MATH1321 -- Trigonometry

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Derivation of the Sum-to-Product Identities

The textbook [1] neglects to derive the Sum-to-Product Identities.

The derivation of the Sum-to-Product Identities follows from the derivation of the Product-to-Sum Identities (pg 237) by means of an appropriate variable substitution.

Product-to-Sum Identities

$\cos A \cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)]$
$\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$
$\sin A \cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$
$\cos A \sin B = \frac{1}{2}[\sin(A+B) - \sin(A-B)]$

Now, perform a change of variables: $\begin{aligned} M &= A + B \\ N &= A - B \end{aligned}$

Solve for A by adding the two equations: $M + N = 2A \Rightarrow A = \left(\frac{M+N}{2}\right)$

Solve for B by subtracting the two equations: $M - N = 2B \Rightarrow B = \left(\frac{M-N}{2}\right)$

Finally, plug in these expressions for A and B into the first Product-to-Sum Identity, and multiply both sides of each identity by 2:

$$\cos A \cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)]$$
$$\Rightarrow \cos\left(\frac{M+N}{2}\right)\cos\left(\frac{M-N}{2}\right) = \frac{1}{2}[\cos M + \cos N]$$
$$\Rightarrow \cos M + \cos N = 2\cos\left(\frac{M+N}{2}\right)\cos\left(\frac{M-N}{2}\right)$$

Repeating this process for the other 3 identities will yield the remaining Sum-to-Product identities (pg 238):

$\sin M + \sin N = 2\sin\left(\frac{M+N}{2}\right)\cos\left(\frac{M-N}{2}\right)$
$\sin M - \sin N = 2\cos\left(\frac{M+N}{2}\right)\sin\left(\frac{M-N}{2}\right)$
$\cos M + \cos N = 2\cos\left(\frac{M+N}{2}\right)\cos\left(\frac{M-N}{2}\right)$
$\cos M - \cos N = -2\sin\left(\frac{M+N}{2}\right)\sin\left(\frac{M-N}{2}\right)$

References

[1] M. L. Lial, J. E. Hornsby, D. Schneider. *Trigonometry*. Pearson, Boston, MA, 9th Edition, 2009.