Texas Tech University
MATH1321 -- Trigonometry

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Vector Normalization

The textbook [1] does not discuss vector normalization.

The need to normalize a vector arises in physics, computer graphics [3], calculus [2], and higher mathematics as some processes are concerned more with a vector's direction than its magnitude.

For the purposes of a plane trigonometry course, only 2-D vectors will be considered. Multivariable calculus [2] looks at 3-D vectors. Linear algebra [4] and higher math courses will investigate vectors of even higher dimensions!

To normalize a vector means to scale it such that its magnitude is one and its direction remains the same.

Given vector $\mathbf{a} = \langle a_1, a_2 \rangle = a_1 \hat{\mathbf{i}} + a_2 \hat{\mathbf{j}} \quad (\text{where } \hat{\mathbf{i}} = \langle 1,0 \rangle,\ \hat{\mathbf{j}} = \langle 0,1 \rangle)$

the normalization of vector **a** is $\hat{\mathbf{a}} = \dfrac{\mathbf{a}}{|\mathbf{a}|} = \dfrac{\langle a_1, a_2 \rangle}{\sqrt{a_1^2 + a_2^2}} \equiv \langle \hat{a}_1, \hat{a}_2 \rangle$

Note that a variable with a 'hat' like $\hat{\mathbf{a}}$ denotes a <u>unit vector</u> (with magnitude 1).

One nice property of unit (normalized) vectors is $\hat{\mathbf{a}} \cdot \hat{\mathbf{a}} = \hat{a}_1 \hat{a}_1 + \hat{a}_2 \hat{a}_2 = (\hat{a}_1)^2 + (\hat{a}_2)^2 = |\hat{\mathbf{a}}|^2 = (1)^2 = 1$

<u>Example (E11):</u> Given vector $\mathbf{u} = \langle -4, 5 \rangle$ normalize it -- that is, find $\hat{\mathbf{u}}$

$$\hat{\mathbf{u}} = \frac{\mathbf{u}}{|\mathbf{u}|} = \frac{1}{\sqrt{(-4)^2 + 5^2}} \langle -4, 5 \rangle = \frac{1}{\sqrt{41}} \langle -4, 5 \rangle = \left\langle -\frac{4}{\sqrt{41}}, \frac{5}{\sqrt{41}} \right\rangle \approx \langle 0.0976, 0.1220 \rangle$$

<u>REMARK:</u> Contrary to intuition, it's better to leave square roots (if they occur) in the denominator.

One can check this solution using the property of unit vectors:

$$\hat{\mathbf{u}} \cdot \hat{\mathbf{u}} = \hat{u}_1 \hat{u}_1 + \hat{u}_2 \hat{u}_2 = (\hat{u}_1)^2 + (\hat{u}_2)^2 = \left(-\frac{4}{\sqrt{41}}\right)^2 + \left(\frac{5}{\sqrt{41}}\right)^2 = \frac{16}{41} + \frac{25}{41} = \frac{16+25}{41} = \frac{41}{41} = 1$$

References

[1] M. L. Lial, J. E. Hornsby, D. Schneider. *Trigonometry*. Pearson, Boston, MA, 9th Edition, 2009.

[2] J. Stewart. *Multivariable Calculus*. Brooks/Cole Publishing, Pacific Grove, CA, 2nd Edition, 1991.

[3] E. Angel, D. Schreiner. *Interactive Computer Graphics: A Top-Down Approach with Shader-Based OpenGL*. Addison Wesley, Upper Saddle River, NJ, 6th Edition, 2011.

[4] G. Strang. *Introduction to Linear Algebra*. Wellesley-Cambridge Press, Wellesley, MA, 3rd Edition, 2003.