TTU - MATH1331 (Business Calculus) - JOSH ENGWER - 11/17/2011

PRACTICE EXAM 3 - SOLUTIONS (Chapter 11 except 11.3)

* For some problems, only the answer and maybe a hint is given – you have to fill in the details!

* Exceptions to the above : Problems 4(a), 4(b), 5, 6, 7, 8, 9

1. a) $\frac{1}{5}x^5 - \frac{3}{5}x^{10/3} - 4x^{-1} - \frac{2}{3}x^{3/2} + C$ or $\frac{1}{5}x^5 - \frac{3}{5}\sqrt[3]{x^{10}} - \frac{4}{x} - \frac{2}{3}\sqrt{x^3} + C$

   b) $\frac{1}{6}(\ln z)^6 + C$ c) $\frac{1}{3}w^3 - \frac{1}{6}w^6 + w - \frac{1}{4}w^4 + C$

2. a) $\frac{3}{2}(\ln 8 - \ln 5) = \frac{3}{2}\ln\left(\frac{8}{5}\right) \approx 0.705005$ b) $e^4 + 3(4^{1/3}) + \frac{2}{7}(4^{-7/2}) - e - \frac{23}{7} \approx 53.35859$

   c) $\frac{8}{15} + \frac{4}{15}\sqrt{2} - 6\sqrt{6} \approx -13.78648$

3. Average value $= \frac{1}{3-0}\int_0^3 \frac{x}{\sqrt{x^2+16}}\,dx = \boxed{\frac{1}{3} \approx 0.33333}$

4. a) Since the solution to equation $f(x) = 0 \iff 3x^2 + 2x + 1 = 0$ has complex (undefined) solutions, and $f(0) = 1 > 0$, the curve $f(x)$ lies totally above the x-axis.
Hence, the usual definite integral will yield the correct area:

Area $= \int_{-1}^{2} (3x^2 + 2x + 1)\,dx = \left[x^3 + x^2 + x\right]_{x=-1}^{x=2} = [(2)^3 + (2)^2 + (2)] - [(-1)^3 + (-1)^2 + (-1)]$
$= (8 + 4 + 2) - (-1 + 1 - 1) = 14 - (-1) = \boxed{15}$

b) First, determine which curve is higher over interval $[1, 3]$: pick some value $c \in (1, 3)$, say $c = 2$
Then, $g(2) = 10 + e^{2(2)} = 10 + e^4 \approx 64.6$, and $h(2) = 4 + \ln 2 \approx 4.69 \implies g(t) \geq h(t)$ on $[1, 3]$

Hence, Area $= \int_1^3 [g(t) - h(t)]\,dt = \int_1^3 [(10 + e^{2t}) - (4 + \ln t)]\,dt = \int_1^3 (6 + e^{2t} - \ln t)\,dt$

$= \left[6t + \frac{1}{2}e^{2t} - (t\ln t - t)\right]_{t=1}^{t=3} = \left(18 + \frac{1}{2}e^6 - 3\ln 3 + 3\right) - \left(6 + \frac{1}{2}e^2 - 1\ln 1 + 1\right)$

$= \boxed{14 - \ln 27 + \frac{1}{2}(e^6 - e^2) \approx 208.724032}$

5. Find points of intersection by solving equation $f(x) = g(x)$ for x :
$f(x) = g(x) \iff x^4 = x \iff x^4 - x = 0 \iff x(x^3 - 1) = 0 \iff x = 0$ or $x^3 - 1 = 0$
$\iff x = 0$ or $x^3 = 1 \iff x = 0$ or $x = 1 \implies$ interval is $[0, 1]$

Now, determine which curve is higher over interval $[0, 1]$: pick some value $c \in (0, 1)$, say $c = \frac{1}{2}$

Then, $f\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^4 = \frac{1}{16} \approx 0.0625$, and $g\left(\frac{1}{2}\right) = \frac{1}{2} = 0.5 \implies g(x) \geq f(x)$ on $[0, 1]$

Hence, Area $= \int_0^1 [g(x) - f(x)]\,dx = \int_0^1 (x - x^4)\,dx = \left[\frac{1}{2}x^2 - \frac{1}{5}x^5\right]_{x=0}^{x=1} = \left(\frac{1}{2} - \frac{1}{5}\right) - (0 - 0) = \boxed{\frac{3}{10}}$

6. First, find $(\bar{x}, \bar{p})$ by solving $D(x) = S(x)$ for x :
$D(x) = S(x) \iff -0.1x^2 - x + 40 = 0.1x^2 + 2x + 20 \iff 0.2x^2 + 3x - 20 = 0 \iff 2x^2 + 30x - 200 = 0$
$\iff x^2 + 15x - 100 = 0 \iff (x+20)(x-5) = 0 \iff x = -20$ or $x = 5$,
but a negative x-value in this context is absurd, so discard $x = -20$.
Hence, $\bar{x} = 5$, now find $\bar{p}$ by computing $D(\bar{x})$ or $S(\bar{x})$, whichever you prefer :
$\bar{p} = S(\bar{x}) = S(5) = 0.1(5)^2 + 2(5) + 20 = 32.5$

Now, find the consumers' surplus : $CS = \int_0^{\bar{x}} [D(x) - \bar{p}]\, dx = \int_0^5 [(-0.1x^2 - x + 40) - (32.5)]\, dx$

$$\implies CS = \int_0^5 (-0.1x^2 - x + 7.5)\, dx = \left[\frac{-0.10}{3}x^3 - \frac{1}{2}x^2 + 7.5x\right]_{x=0}^{x=5} \approx 20.83333$$

Thus, consumers' surplus $= (20.8333)(100) = \boxed{\$2083.33}$

Now, find the suppliers' surplus : $PS = \int_0^{\bar{x}} [\bar{p} - S(x)]\, dx = \int_0^5 [32.5 - (0.1x^2 + 2x + 20)]\, dx$

$$\implies PS = \int_0^5 (-0.1x^2 - 2x + 12.5)\, dx = \left[\frac{-0.10}{3}x^3 + x^2 + 12.5x\right]_{x=0}^{x=5} \approx 33.3333$$

Thus, suppliers' surplus $= (33.3333)(100) = \boxed{\$3333.33}$

REMARK: Why are the surpluses multiplied by 100 at the end?? ANSWER: Read the problem carefully...

7. This is asking for the <u>amount</u> of an <u>annuity</u>, which is the regular deposits to the retirement account.
So, $P = 4000$ dollars/yr, $r = 0.08$, $T = 20$ yrs, and $m = 1$ (since 4000 dollars is deposited <u>once</u> per year)
Hence, $A = \frac{mP}{r}\left(e^{rT} - 1\right) = \frac{(1)(4000)}{(0.08)}\left[e^{(0.08)(20)} - 1\right] = \boxed{\$197,651.62}$

8. This is asking for the <u>present value</u> of an <u>income stream</u>.
So, $R(t) = 80,000$ dollars/yr, $r = 0.10$, $T = 10$ yrs
Hence, $PV = \int_0^T R(t)e^{-rt}\, dt = \int_0^{10} 80,000e^{-0.10t}\, dt = 80,000 \int_0^{10} e^{-0.10t}\, dt$

REMARK: The variable t is the variable of integration – do not confuse t with T !

This integral requires <u>substitution</u>:
Let $u = -0.10t$, then $\frac{du}{dt} = [-0.10t]' = -0.10 \iff du = -0.10dt \iff dt = -\frac{1}{0.10}du$

Don't forget to update the limits of integration: $u(10) = -0.10(10) = -1$, $u(0) = -0.10(0) = 0$

So, $PV = 80,000 \int_0^{10} e^{-0.10t}\, dt = 80,000 \int_0^{-1} e^u \left(-\frac{1}{0.10}du\right) = \frac{80,000}{0.10}\int_{-1}^{0} e^u\, du = 800,000[e^u]_{u=-1}^{u=0}$
$= 800,000[e^0 - e^{-1}] = 800,000[1 - e^{-1}] \approx \boxed{\$505,696.45}$

REMARK: Why did the negative sign vanish during the integration?
Because the 'flip interval' rule was used (see the Supplementary Notes about definite integrals.)

9. a) $L(0.3) = \frac{17}{18}(0.3)^2 + \frac{1}{18}(0.3) \approx \boxed{0.1017}$, $L(0.6) = \frac{17}{18}(0.6)^2 + \frac{1}{18}(0.6) \approx \boxed{0.3733}$

b) $L(0.3) = 0.1017$ means that the poorest 30% of the country receives 10.17% of the total income
$L(0.6) = 0.3733$ means that the poorest 60% of the country receives 37.33% of the total income

c) $G = 2\int_0^1 [x - L(x)]\, dx = 2\int_0^1 \left[x - \left(\frac{17}{18}x^2 + \frac{1}{18}x\right)\right]\, dx = 2\int_0^1 \left(-\frac{17}{18}x^2 + \frac{17}{18}x\right)\, dx$
$= (2)\left(\frac{17}{18}\right)\int_0^1 (-x^2 + x)\, dx = \frac{17}{9}\left[-\frac{1}{3}x^3 + \frac{1}{2}x^2\right]_{x=0}^{x=1} = \frac{17}{9}\left(-\frac{1}{3} + \frac{1}{2}\right) = \frac{17}{9}\left(\frac{1}{6}\right) = \frac{17}{54} \approx \boxed{0.3148}$

BONUS QUESTIONS:

(B1) DEFINITION: A function f is even if $f(-x) = f(x)$. A function f is odd if $f(-x) = -f(x)$

a) f is odd b) g is even c) h is neither even nor odd d) φ is neither even nor odd

Show the appropriate work to justify your choice of even, odd, or neither.

Here's the work for part (a) :

$$f(-x) = 2(-x)^{59} - \frac{1}{7}(-x)^{11} - (-x) = 2(-1)^{59}(x^{59}) - \frac{1}{7}(-1)^{11}(x^{11}) + x$$

$$= 2(-1)x^{59} - \frac{1}{7}(-1)x^{11} + x = -2x^{59} + \frac{1}{7}x^{11} + x = -\left(2x^{59} - \frac{1}{7}x^{11} - x\right) = -f(x) \implies f \text{ is odd}$$

(B2) ???? (Come by my office hours and tell me what your answer is.)