CONSTRAINED OPTIMIZATION

STEP 1: Identify & label all **relevant variables**.

STEP $1^1/_2$: If applicable, **draw a picture**.

STEP 2: Identify the **multivariable function to be optimized**.

STEP 3: Identify all **constraints**. Expect some constraints to be **equations**, others to be **inequalities**.

Some will be blatantly obvious, others not so much...

STEP $3^1/_2$: Solve each **equality constraint** in terms of the **common variable**. [e.g. x]

If there are **several common variables**, pick the **easiest to work with**.

Then, write each **inequality constraint** in terms of the **common variable**.

STEP 4: Use the **equality constraint(s)** to rewrite the function **in terms of only ONE variable**. [e.g. $f(x)$]

STEP 5: Find the **practical domain** of f. PractDom(f) := Dom$(f) \cap$ (each **inequality constraint** involving only x)

STEP 6: Pose the optimization problem in terms of $f(x)$, absolute extrema, PractDom(f).

STEP 7: If PractDom(f) is **closed & bounded**, that is, PractDom$(f) = [a, b]$, where $-\infty < a < b < \infty$, then:

Abs max value of f over $[a, b] = \max\{f(a), f(b), f(c_1), f(c_2), \ldots, f(c_k)\}$, where $c_1, \ldots, c_k$ are **critical #'s** of f.

Abs max of f over $[a, b]$ is the corresponding x-value to the abs max value.

Abs min value of f over $[a, b] = \min\{f(a), f(b), f(c_1), f(c_2), \ldots, f(c_k)\}$, where $c_1, \ldots, c_k$ are **critical #'s** of f.

Abs min of f over $[a, b]$ is the corresponding x-value to the abs min value.

STEP 8: If PractDom(f) is **NOT closed & bounded**, use **Curve Sketching** techniques. (See [CURVE-SKETCH-II])

REMARK: This is the most general procedure for constrained optimization problems.

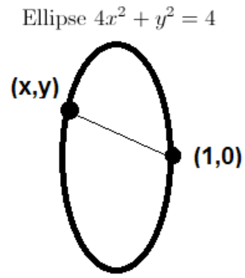
It's possible (but not likely) that the picture, practical domain, or function to optimize may be given a priori.

EXAMPLE: Find the points on the ellipse $4x^2 + y^2 = 4$ that are farthest away from the point $(1, 0)$.

What is the maximum distance possible?

STEP 1: Let (x, y) denote the point(s) on the ellipse farthest from the point $(1, 0)$.

STEP $1^1/_2$: Draw an appropriate picture:



STEP 2: Maximizing distance is equivalent to maximizing the **square of the distance**.

This spares us from dealing with a square root (and the Chain Rule later on).

Hence, let $D(x, y) = (x - 1)^2 + (y - 0)^2 = \left(x^2 - 2x + 1\right) + y^2$

STEP 3: The one constraint is that the point(s) must lie on the ellipse. Thus the equality constraint is $4x^2 + y^2 = 4$.

STEP $3^1/_2$: It's slightly easier to solve for y : $y = \pm\sqrt{4 - 4x^2}$

STEP 4: $f(x) = D\left(x, \pm\sqrt{4 - 4x^2}\right) = x^2 - 2x + 1 + \left(\pm\sqrt{4 - 4x^2}\right)^2 = x^2 - 2x + 1 + 4 - 4x^2 = -3x^2 - 2x + 5$

STEP 5: $\text{PractDom}(f) = \text{Dom}(f) = [-1, 1]$ $\quad$ $\Big($Why is $\text{Dom}(f) = [-1, 1]$ instead of $\mathbb{R}$???$\Big)$

STEP 6: "Find the absolute maximum of $f(x)$ over the interval $[-1, 1]$."

STEP 7: Observe that $\text{PractDom}(f) = [-1, 1]$ is closed & bounded.

$f'(x) = \dfrac{d}{dx}\left[-3x^2 - 2x + 5\right] = -6x - 2 \overset{set}{=} 0 \implies x = -\dfrac{1}{3}$ is the only critical number of f.

Absolute max value of f over $[-1, 1] = \max\left\{f(-1), f(1), f\left(-\frac{1}{3}\right)\right\} = \max\left\{4, 0, \frac{16}{3}\right\} = \frac{16}{3}$

$\implies$ absolute max occurs at $x = -\dfrac{1}{3} \implies y = \pm\sqrt{4 - 4\left(-\frac{1}{3}\right)^2} = \pm\dfrac{\sqrt{32}}{3}$

Therefore, the two points on the ellipse farthest from the point $(1, 0)$ are $\left(-\dfrac{1}{3}, -\dfrac{\sqrt{32}}{3}\right)$ and $\left(-\dfrac{1}{3}, \dfrac{\sqrt{32}}{3}\right)$

Remember, $f(x)$ is modeling the **square** of the distance, hence:

Maximum distance possible is the **square-root** of the abs max value of f over $[-1, 1] = \sqrt{\dfrac{16}{3}} = \dfrac{4}{\sqrt{3}} \approx 2.3094$

Therefore, the maximum distance possible is $\dfrac{4}{\sqrt{3}} \approx 2.3094$