CURVE SKETCHING (I): HILLS, VALLEYS, AND INFLECTION POINTS

EXAMPLE: Let $g(t) = 3t^4 - 2t^3 - 12t^2 + 18t - 5$

(a) Identify the **domain** of g.

$\text{Dom}(g) = \text{Dom}\left(3t^4 - 2t^3 - 12t^2 + 18t - 5\right) = \boxed{\mathbb{R} = (-\infty, \infty)}$

(b) Find all **critical numbers** of g which lie in the **interior** of its **domain** $\implies$ $\text{int}\,(\text{Dom}(g)) = \text{int}\,((-\infty,\infty)) = (-\infty,\infty)$

$g'(t) = \dfrac{d}{dx}\left[3t^4 - 2t^3 - 12t^2 + 18t - 5\right] \overset{D.5}{=} 12t^3 - 6t^2 - 24t + 18$

$g'(t) \overset{set}{=} 0 \implies 12t^3 - 6t^2 - 24t + 18 = 0 \implies 6\left(2t^3 - t^2 - 4t + 3\right) = 0 \implies 2t^3 - t^2 - 4t + 3 = 0$

Now, this **cubic polynomial** is not obvious to factor since **factoring by grouping** does not work here.

So, since the cubic polynomial has **all integer coefficients**, try using the **Rational Roots Theorem** :

Possible rational solutions are $\pm\frac{3}{1}, \pm\frac{3}{2}, \pm\frac{1}{1}, \pm\frac{1}{2}$

Try $t = 1 \implies g'(1) = 12 - 6 - 24 + 18 = 0 \implies t = 1$ solves the equation $g'(t) = 0 \implies (t-1)$ factors the cubic.

Now, factor out $(t-1)$ via **polynomial division** or **synthetic division**:

$g'(t) = 0 \implies 2t^3 - t^2 - 4t + 3 = 0 \implies (t-1)(2t^2 + t - 3) = 0 \implies (t-1)(2t+3)(t-1) = 0 \implies t \in \left\{-\frac{3}{2}, 1\right\}$

Now, consider where $g'(t)$ is undefined yet $g(t)$ is defined: g' is a **polynomial** $\implies$ g' is undefined nowhere.

Thus, $\boxed{\text{the critical numbers of } g \text{ are } t \in \left\{-\dfrac{3}{2}, 1\right\}}$

(c) Identify all **relative minima, relative maxima** as well as the interval(s) where g **increases** and **decreases**.

In order to identify all this, **build the slope table** for g:

t	-10	$-\frac{3}{2}$	0	1	10
$g'(t)$	$-$	0	$+$	0	$+$
slope	\	$-$	/	$-$	/

$\implies$ $\boxed{g \text{ has a relative min (valley) at } t = -\dfrac{3}{2}}$ and $t = 1$ is NOT a relative extremum.

Relative extrema separate intervals of increasing/decreasing:

$\boxed{g \text{ increases over } \left(-\dfrac{3}{2}, \infty\right) \text{ OR } \left(-\dfrac{3}{2}, 1\right) \cup (1,\infty),\ g \text{ decreases over } \left(-\infty, -\dfrac{3}{2}\right)}$

(d) Find all **CFIP's** (candidates for inflection point) of g which lie in the **interior** of its **domain**.

$g''(t) = \dfrac{d}{dt}\left[12t^3 - 6t^2 - 24t + 18\right] \overset{D.5}{=} 36t^2 - 12t - 24$

$g''(t) \overset{set}{=} 0 \implies 36t^2 - 12t - 24 = 0 \implies 12\left(3t^2 - t - 2\right) = 0 \implies 3t^2 - t - 2 = 0 \implies (3t+2)(t-1) = 0 \implies t \in \left\{-\frac{2}{3}, 1\right\}$

Now, consider where $g''(t)$ is undefined yet $g(t)$ is defined: g'' is a **polynomial** $\implies$ g'' is undefined nowhere.

Thus, $\boxed{\text{the CFIP's of } g \text{ are } t \in \left\{-\dfrac{2}{3}, 1\right\}}$

(e) Identify all **inflection points** as well as the interval(s) where g is **concave up** and **concave down**.

In order to identify all this, **build the concavity table** for g:

t	-10	$-\frac{2}{3}$	0	1	10
$g''(t)$	$+$	0	$-$	0	$+$
concavity	$\cup$	$*$	$\cap$	$*$	$\cup$

$\implies$ $\boxed{g \text{ inflection points at } t \in \left\{-\dfrac{2}{3}, 1\right\}}$

Inflection points separate intervals of concave up/concave down:

$\boxed{g \text{ is concave up (smiles) over } \left(-\infty, -\dfrac{2}{3}\right) \cup (1,\infty),\ g \text{ is concave down (frowns) over } \left(-\dfrac{2}{3}, 1\right)}$

(f) Sketch the graph of g.

Find the **y-coordinate** of every **relative min, relative max, and inflection point**. Then, **plot those points**.

Next, find the **y-intercept** by finding $b = g(0)$, then **plot the y-intercept** $(0, b)$.

If it's feasible, find the **x-intercept** by solving $g(a) = 0$ for a, then **plot the x-intercept** $(a, 0)$.

Finally, complete the curve sketch using the guidance of the **third rows** of the **slope** & **concavity tables** of g.

EXAMPLE: Let $r(\theta) = 3\sin\theta + \cos\theta$, where $\theta \in [0, 2\pi)$.

(a) Identify the **domain** of r.

$\text{Dom}(r) = \text{Dom}\left(3\sin\theta + \cos\theta\right) \cap [0, 2\pi) \overset{DM.S}{=} [\text{Dom}(\sin\theta) \cap \text{Dom}(\cos\theta)] \cap [0, 2\pi) \overset{DM.10}{=} \mathbb{R} \cap \mathbb{R} \cap [0, 2\pi) = \boxed{[0, 2\pi)}$

(b) Find all **critical numbers** of r which lie in the **interior** of its **domain** $\Longrightarrow$ $\text{int}\left(\text{Dom}(r)\right) = \text{int}\left([0, 2\pi)\right) = (0, 2\pi)$

$r'(\theta) = \dfrac{d}{d\theta}[3\sin\theta + \cos\theta] \overset{D.2}{=} \dfrac{d}{d\theta}[3\sin\theta] + \dfrac{d}{d\theta}[\cos\theta] = 3\cos\theta - \sin\theta$

$r'(\theta) \overset{set}{=} 0 \implies 3\cos\theta - \sin\theta = 0 \implies \cos\theta\left(3 - \frac{\sin\theta}{\cos\theta}\right) = \cos\theta\,(3 - \tan\theta) = 0 \implies \cos\theta = 0$ OR $\tan\theta = 3$

$\cos\theta = 0 \implies \theta \in \left\{\cdots, -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \cdots\right\} \cap [0, 2\pi) = \left\{\frac{\pi}{2}, \frac{3\pi}{2}\right\}$

$\tan\theta = 3 \implies \theta = \arctan(3) \overset{calc}{\approx} 1.2490$ (radians). Now, since $\tan\theta > 0$ and $\theta \in [0, 2\pi)$, θ must lie in Quadrants I & III.

Sure enough, 1.2490 (radians) $\in$ QI $= \left(0, \frac{\pi}{2}\right) \approx (0, 1.57)$, but what is the corresponding angle in QIII??

Use reference angles : $\theta_r = 1.2490 \in$ QI $\approx (0, 1.57) \implies \theta = \pi + 1.2490 \approx 4.3906 \in$ QIII $= \left(\pi, \frac{3\pi}{2}\right) \approx (3.14, 4.71)$

Now, check whether each solution satisfies the **original equation** $r'(\theta) = 3\cos\theta - \sin\theta = 0$:

$r'\left(\frac{\pi}{2}\right) = 0 - 1 = -1 \neq 0 \implies$ discard $\theta = \frac{\pi}{2}$ $\quad r'\left(\frac{3\pi}{2}\right) = 0 + 1 = 1 \neq 0 \implies$ discard $\theta = \frac{3\pi}{2}$

$r'(1.2490) = 0.0001447 \approx 0 \implies \theta \approx 1.2490$ is indeed a solution.

$r'(4.3906) = -0.0001215 \approx 0 \implies \theta \approx 4.3906$ is indeed a solution.

Thus, the **critical numbers** of r are $\theta \in \{1.2490, 4.3906\}$

(c) Identify all **relative minima, relative maxima** as well as the interval(s) where r **increases** and **decreases**.

In order to identify all this, **build the slope table** for r:

θ	0	$\frac{\pi}{4} \approx 0.785$	1.2490	$\pi \approx 3.14$	4.3906	$\frac{3\pi}{2} \approx 4.71$	$2\pi \approx 6.28$
$r'(\theta)$	##	+	0	−	0	+	##
slope	##	/	–	\	–	/	##

Thus, interpreting the 'slope' row reveals: Rel. max (hill) at $\theta \approx 1.2490$ and rel. min (valley) at $\theta \approx 4.3906$

$\implies$ r increases over $(0, 1.2490) \cup (4.3906, 2\pi)$, r decreases over $(1.2490, 4.3906)$

(d) Find all **CFIP's** (candidates for inflection point) of r which lie in the **interior** of its **domain**.

$r''(\theta) = \dfrac{d}{d\theta}[3\cos\theta - \sin\theta] \overset{D.2}{=} \dfrac{d}{d\theta}[3\cos\theta] - \dfrac{d}{d\theta}[\sin\theta] = -3\sin\theta - \cos\theta$

$r''(\theta) \overset{set}{=} 0 \implies -3\sin\theta - \cos\theta = 0 \implies -\cos\theta\left(3\frac{\sin\theta}{\cos\theta} + 1\right) = 0 \implies \cos\theta = 0$ OR $\tan\theta = -\frac{1}{3}$

$\tan\theta = -\frac{1}{3} \implies \theta = \arctan\left(-\frac{1}{3}\right) \overset{calc}{\approx} -0.32175$. Now, since $\tan\theta < 0$ and $\theta \in [0, 2\pi)$, θ must lie in Quadrants II & IV.

But $-0.32175 \notin [0, 2\pi)$, so find the **coterminal angle:** $\theta = -0.32175 + 2\pi \approx 5.96144 \in$ QIV $= \left(\frac{3\pi}{2}, 2\pi\right) \approx (4.71, 6.28)$

As for QII, use **reference angles** : $\theta_r = 0.32175 \in$ QI $\approx (0, 1.57) \implies \theta = \pi - \theta_r \approx 2.81984 \in$ QII $= \left(\frac{\pi}{2}, \pi\right) \approx (1.57, 3.14)$

Now, check whether each solution satisfies the **original equation** $r''(\theta) = -3\sin\theta - \cos\theta = 0$:

$r''\left(\frac{\pi}{2}\right) = -3 \neq 0$ and $r''\left(\frac{3\pi}{2}\right) = 3 \neq 0$, so **discard** $\theta = \frac{\pi}{2}$ and $\theta = \frac{3\pi}{2}$

$r''(2.81984) = -0.00000664 \approx 0$ and $r''(5.96144) = -0.00001659 \approx 0$

Thus, the CFIP's of r are $\theta \in \{2.81984, 5.96144\}$

(e) Identify all **inflection points** as well as the interval(s) where r is **concave up** and **concave down**.

In order to identify all this, **build the concavity table** for r:

θ	0	$\frac{\pi}{2} \approx 1.57$	2.81984	$\pi \approx 3.14$	5.96144	6	$2\pi \approx 6.28$
$r''(\theta)$	##	−	0	+	0	−	##
concavity	##	∩	*	∪	*	∩	##

$\implies$ r has inflection points at $\theta \in \{2.81984, 5.96144\}$

$\implies$ r is concave up (smiles) over $(2.81984, 5.96144)$, r is concave down (frowns) over $(0, 2.81984) \cup (5.96144, 2\pi)$

(f) Sketch the graph of r.

(See the part (f) of the previous example)

EXAMPLE: Let $f(x) = 2x + \arccos x$

(a) Identify the **domain** of f.

$$\text{Dom}(f) \overset{DM.S}{=} \text{Dom}(2x) \cap \text{Dom}(\arccos x) \overset{\substack{DM.1\\DM.V}}{=} \mathbb{R} \cap \text{Rng}(\cos x) \overset{RG.4}{=} \mathbb{R} \cap [-1,1] = \boxed{[-1,1]}$$

(b) Find all **critical numbers** of f which lie in the **interior** of its **domain** $\Longrightarrow$ $\text{int}\,(\text{Dom}(f)) = \text{int}\,([-1,1]) = (-1,1)$

$$f'(x) = \frac{d}{dx}[2x + \arccos x] \overset{D.2}{=} \frac{d}{dx}[2x] + \frac{d}{dx}[\arccos x] \overset{D.5}{=} 2 + \frac{d}{dx}[\arccos x] \overset{D.17}{=} 2 - \frac{1}{\sqrt{1-x^2}}$$

$$f'(x) \overset{set}{=} 0 \implies 2 - \frac{1}{\sqrt{1-x^2}} = 0 \implies 2\sqrt{1-x^2} - 1 = 0 \implies \sqrt{1-x^2} = \frac{1}{2} \implies 1 - x^2 = \frac{1}{4} \implies x = \pm\frac{\sqrt{3}}{2} \approx \pm 0.8660$$

Now, consider where $f'(x)$ is undefined yet $f(x)$ is defined: $f'(x)$ is undefined $\implies \sqrt{1-x^2} = 0 \implies x = \pm 1$

$f(1) = 2(1) + \arccos 1 = 2 + 0 = 2,\quad f(-1) = 2(-1) + \arccos(-1) = -2 + \pi = \pi - 2 \implies$ both $f(1), f(-1)$ are defined.

But $x = \pm 1$ lie on the **boundary** of the **domain** of f, so discard these two values.

Thus, $\boxed{\text{the critical numbers of } f \text{ are } x \in \left\{-\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}\right\}}$

(c) Identify all **relative minima, relative maxima** as well as the interval(s) where f **increases** and **decreases**.

In order to identify all this, **build the slope table** for f:

x	-1	-0.9	-0.8660	0	0.8660	0.9	1
$f'(x)$	##	$-$	0	$+$	0	$-$	##
slope	##	\	–	/	–	\	##

Thus, interpreting the 'slope' row reveals: $\boxed{f \text{ has relative min (valley) at } x = -0.8660, f \text{ has relative max (hill) at } x = 0.8660}$

Relative extrema separate intervals of increasing/decreasing:

$\boxed{f \text{ increases over } (-0.8660, 0.8660),\ f \text{ decreases over } [-1, -0.8660) \cup (0.8660, 1]}$

(d) Find all **CFIP's** (candidates for inflection point) of f which lie in the **interior** of its **domain**.

$$f''(x) = \frac{d}{dx}\left[2 - \frac{1}{\sqrt{1-x^2}}\right] = 0 + \frac{d}{dx}\left[-\left(1-x^2\right)^{-1/2}\right] \overset{D.22}{=} \frac{1}{2}\left(1-x^2\right)^{-3/2}(-2x) = -\frac{x}{\sqrt{(1-x^2)^3}}$$

$$f''(x) \overset{set}{=} 0 \implies -\frac{x}{\sqrt{(1-x^2)^3}} = 0 \implies x = 0$$

Now, consider where $f''(x)$ is undefined: $f''(x)$ is undefined $\implies \sqrt{(1-x^2)^3} = 0 \implies x = \pm 1$

But $x = \pm 1$ lie on the **boundary** of the **domain** of f, so discard these two values.

Thus, $\boxed{\text{the CFIP's of } f \text{ are } x \in \{0\}}$

(e) Identify all **inflection points** as well as the interval(s) where f is **concave up** and **concave down**.

In order to identify all this, **build the concavity table** for f:

x	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1
$f''(x)$	##	$+$	0	$-$	##
concavity	##	$\cup$	*	$\cap$	##

Thus, interpreting the 'concavity' row reveals: $\boxed{f \text{ has inflection point at } x = 0}$

Inflection points separate intervals of concave up/concave down:

$\boxed{f \text{ is concave up (smiles) over } [-1, 0), f \text{ is concave down (frowns) over } (0, 1]}$

(f) Sketch the graph of f.

Find the **y-coordinate** of every **relative min, relative max, and inflection point**. Then, **plot those points**.

Next, find the **y-intercept** by finding $b = f(0)$, then **plot the y-intercept** $(0, b)$.

If it's feasible, find the **x-intercept** by solving $f(a) = 0$ for a, then **plot the x-intercept** $(a, 0)$.

Finally, complete the curve sketch using the guidance of the **third rows** of the **slope** & **concavity tables** of f.