DERIVATIVES

DEFINITION OF THE DERIVATIVE: $f'(x) := \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$

- **Geometric interpretation** of $f'(x)$: **Slope of the tangent line** to the curve f at x.
- **Application interpretation** of $f'(x)$: **(Instantaneous) rate of change** of f with respect to x.

VARIOUS NOTATIONS FOR DERIVATIVES: Assume $y = f(x)$. Then:

- 1^{st} (order) derivative of f with respect to x : $f'(x), y', \frac{dy}{dx}, Df$
- 2^{nd} (order) derivative of f with respect to x : $f''(x), y'', \frac{d^2y}{dx^2}, D^2f$
- 3^{rd} (order) derivative of f with respect to x : $f'''(x), y''', \frac{d^3y}{dx^3}, D^3f$
- 4^{th} (order) derivative of f with respect to x : $f^{(4)}(x), y^{(4)}, \frac{d^4y}{dx^4}, D^4f$

  $\vdots$

- n^{th} (order) derivative of f with respect to x : $f^{(n)}(x), y^{(n)}, \frac{d^ny}{dx^n}, D^nf$

DERIVATIVE RULES: Here, $k \in \mathbb{R}$ and $a \in \mathbb{R}_+ \setminus \{1\}$

- (D.0) (Constant Rule) $\frac{d}{dx}[k] = 0$
- (D.1) (Multiple Rule) $\frac{d}{dx}[kf(x)] = kf'(x)$
- (D.2) (Sum/Diff Rule) $\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)$
- (D.3) (Product Rule) $\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x)$
- (D.4) (Quotient Rule) $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$ "Lo-D-Hi minus Hi-D-Lo all over Lo-squared."
- (D.5) (Power Rule) $\frac{d}{dx}\left[x^k\right] = kx^{k-1}$ (Applies to **roots** & **reciprocal powers** upon rewriting them as powers)
- (D.6) (Sine Rule) $\frac{d}{dx}[\sin x] = \cos x$ (D.7) (Cosine Rule) $\frac{d}{dx}[\cos x] = -\sin x$
- (D.8) (Tangent Rule) $\frac{d}{dx}[\tan x] = \sec^2 x$ (D.9) (Cotangent Rule) $\frac{d}{dx}[\cot x] = -\csc^2 x$
- (D.10) (Secant Rule) $\frac{d}{dx}[\sec x] = \sec x \tan x$ (D.11) (Cosecant Rule) $\frac{d}{dx}[\csc x] = -\csc x \cot x$
- (D.12) (Natural Exp) $\frac{d}{dx}[e^x] = e^x$ (D.13) (Natural Log) $\frac{d}{dx}[\ln x] = \frac{1}{x}$
- (D.14) (General Exp) $\frac{d}{dx}[a^x] = (\ln a)a^x$ (D.15) (General Log) $\frac{d}{dx}[\log_a x] = \frac{1}{(\ln a)} \cdot \frac{1}{x}$
- (D.16) (Inv Sine) $\frac{d}{dx}[\arcsin x] = \frac{1}{\sqrt{1-x^2}}$ (D.17) (Inv Cosine) $\frac{d}{dx}[\arccos x] = -\frac{1}{\sqrt{1-x^2}}$
- (D.18) (Inv Tangent) $\frac{d}{dx}[\arctan x] = \frac{1}{1+x^2}$ (D.19) (Inv Cotangent) $\frac{d}{dx}[\text{arccot } x] = -\frac{1}{1+x^2}$
- (D.20) (Inv Secant) $\frac{d}{dx}[\text{arcsec } x] = \frac{1}{|x|\sqrt{x^2-1}}$ (D.21) (Inv Cosecant) $\frac{d}{dx}[\text{arccsc } x] = -\frac{1}{|x|\sqrt{x^2-1}}$
- (D.22) (Chain Rule - usual form) $\frac{d}{dx}[(f \circ g)(x)] = \frac{d}{dx}[f(g(x))] = f'(g(x))g'(x)$
- (D.23) (Chain Rule - Leibniz form) $v = f(u), u = g(x) \implies \frac{dv}{dx} = \frac{dv}{du} \cdot \frac{du}{dx}$
- (I-DIFF) (Implicit Differentiation) Use to differentiate **implicit functions** and derive the inverse trig rules.
- (L-DIFF) (Logarithmic Differentiation) Use to differentiate overly complicated products, quotients, and exponentials.

EXAMPLE: Let $f(x) = 10 - 2x + \frac{3}{4}x^4 - 4\sqrt[3]{x} + \frac{5}{x^3}$. Find $f'(x)$.

First, rewrite all roots & reciprocal powers as powers : $f(x) = 10 - 2x + \frac{3}{4}x^4 - 4x^{1/3} + 5x^{-3}$. Then:

$$f'(x) = \frac{d}{dx}\left[10 - 2x + \frac{3}{4}x^4 - 4x^{1/3} + 5x^{-3}\right] \overset{D.2}{=} \frac{d}{dx}[10] - \frac{d}{dx}[2x] + \frac{d}{dx}\left[\frac{3}{4}x^4\right] - \frac{d}{dx}\left[4x^{1/3}\right] + \frac{d}{dx}\left[5x^{-3}\right]$$

$$\overset{D.1}{=} \frac{d}{dx}[10] - 2\frac{d}{dx}[x] + \frac{3}{4}\frac{d}{dx}\left[x^4\right] - 4\frac{d}{dx}\left[x^{1/3}\right] + 5\frac{d}{dx}\left[x^{-3}\right] \overset{D.5}{=} \frac{d}{dx}[10] - 2(1) + \frac{3}{4}(4x^3) - 4\left(\frac{1}{3}x^{-2/3}\right) + 5(-3x^{-4})$$

$$\overset{D.0}{=} (0) - 2 + 3x^3 - \frac{4}{3}x^{-2/3} - 15x^{-4}$$ Thus, $\boxed{f'(x) = -2 + 3x^3 - \frac{4}{3}x^{-2/3} - 15x^{-4}}$

EXAMPLE: Let $w = \sin t \cos t - e^t$. Find $\frac{dw}{dt}$.

$$\frac{dw}{dt} = \frac{d}{dt}\left[\sin t \cos t - e^t\right] \overset{D.2}{=} \frac{d}{dt}[\sin t \cos t] - \frac{d}{dt}\left[e^t\right] \overset{D.12}{=} \frac{d}{dt}[\sin t \cos t] - \left(e^t\right) \overset{D.3}{=} \left(\cos t \frac{d}{dt}[\sin t] + \sin t \frac{d}{dt}[\cos t]\right) - e^t$$

$$\overset{D.6}{=} \cos t(\cos t) + \sin t \frac{d}{dt}[\cos t] - e^t \overset{D.7}{=} \cos^2 t + \sin t(-\sin t) - e^t = \cos^2 t - \sin^2 t - e^t$$

Thus, $\boxed{\frac{dw}{dt} = \cos^2 t - \sin^2 t - e^t}$

EXAMPLE: Let $h(y) = (3y+2)^{100}$. Find $h'(y)$.

In anticipation of use of the Chain Rule (usual form), decompose h : $h(y) = f(g(y))$, where $g(y) = 3y + 2$ and $f(y) = y^{100}$

Then: $h'(y) = \frac{d}{dy}\left[(3y+2)^{100}\right] \overset{D.22}{=} 100(3y+2)^{99}\frac{d}{dy}[3y+2] \overset{D.5}{=} (3)(100)(3y+2)^{99} = 300(3y+2)^{99}$

Thus, $\boxed{h'(y) = 300(3y+2)^{99}}$

EXAMPLE: Let $r = \ln\left(\theta^2 + \theta + 1\right)$. Find $\frac{dr}{d\theta}$.

$$\frac{dr}{d\theta} = \frac{d}{d\theta}\left[\ln\left(\theta^2 + \theta + 1\right)\right] \overset{D.22}{=} \frac{1}{\theta^2 + \theta + 1} \cdot \frac{d}{d\theta}\left[\theta^2 + \theta + 1\right] \overset{D.5}{=} \frac{2\theta + 1 + 0}{\theta^2 + \theta + 1}$$ Thus, $\boxed{\frac{dr}{d\theta} = \frac{2\theta + 1}{\theta^2 + \theta + 1}}$

EXAMPLE: Let $z = 2y^3 + y^2 - 1$ and $y = 1 + 5e^x$. Find $\frac{dz}{dx}$.

$$\frac{dz}{dy} = \frac{d}{dy}\left[2y^3 + y^2 - 1\right] \overset{D.5}{=} 6y^2 + 2y \qquad \frac{dy}{dx} = \frac{d}{dx}\left[1 + 5e^x\right] \overset{D.2}{=} 0 + \frac{d}{dx}\left[5e^x\right] \overset{D.12}{=} 5e^x$$

$$\frac{dz}{dx} \overset{D.23}{=} \frac{dz}{dy} \cdot \frac{dy}{dx} = \left(6y^2 + 2y\right)\left(5e^x\right)$$

At this point, it may be tempting to stop and box the previous expression, but this is not correct! Why not?????

The answer lies in reading $\frac{dz}{dx}$ out loud: 'The derivative of z **with respect to** x'

In other words, the derivative must be **completely** in terms of x:

$$\frac{dz}{dx} = \left(6y^2 + 2y\right)\left(5e^x\right) = \left[6(1 + 5e^x)^2 + 2(1 + 5e^x)\right]\left(5e^x\right) = 30e^x(1 + 10e^x + 25e^{2x}) + 10e^x(1 + 5e^x) = 750e^{3x} + 350e^{2x} + 40e^x$$

Thus, $\boxed{\frac{dz}{dx} = 750e^{3x} + 350e^{2x} + 40e^x}$

EXAMPLE: Let $y = x^{\sqrt[3]{x}}$. Find y'.

Since function is an exponential with a variable base and variable power, use logarithmic differentiation:

$\Longrightarrow \ln y = \sqrt[3]{x}\ln x \overset{L.DIFF}{\Longrightarrow} \frac{d}{dx}[\ln y] = \frac{d}{dx}[\sqrt[3]{x}\ln x]$. Let's now differentiate each side individually:

$$\text{RHS} = \frac{d}{dx}[\sqrt[3]{x}\ln x] \overset{D.6}{=} \ln x\frac{d}{dx}\left[x^{1/3}\right] + x^{1/3}\frac{d}{dx}[\ln x] \overset{D.5}{=} \ln x\left(\frac{1}{3}x^{-2/3}\right) + x^{1/3}\frac{d}{dx}[\ln x] \overset{D.13}{=} \frac{1}{3}x^{-2/3}\ln x + x^{1/3}\frac{1}{x}$$

$$\text{LHS} = \frac{d}{dx}[\ln y] \overset{D.22}{=} \frac{1}{y} \cdot \frac{d}{dx}[y] \overset{I-DIFF}{=} \frac{y'}{y}$$

Hence, $\frac{d}{dx}[\ln y] = \frac{d}{dx}[\sqrt[3]{x}\ln x] \Longrightarrow \frac{y'}{y} = \frac{1}{3}x^{-2/3}\ln x + x^{-2/3}$

Thus, $y' = y\left[\frac{1}{3}x^{-2/3}\ln x + x^{-2/3}\right] = \left[x^{\sqrt[3]{x}}\right]\left[\frac{1}{3}x^{-2/3}\ln x + x^{-2/3}\right] \Longrightarrow \boxed{y' = \frac{x^{\sqrt[3]{x}}}{3x^{2/3}}\left[3 + \ln x\right]}$

EXAMPLE: Let $f(x) = x^3$. Find : (a) $f'(x)$ (b) $f''(x)$ (c) $f'''(x)$ (d) slope of tangent line to curve f at $x = -2$

(a) $f'(x) = \frac{d}{dx}[f(x)] = \frac{d}{dx}\left[x^3\right] \overset{D.5}{=} 3x^2$ Hence, $\boxed{f'(x) = 3x^2}$

(b) $f''(x) = \frac{d}{dx}[f'(x)] = \frac{d}{dx}\left[3x^2\right] \overset{D.5}{=} 6x$ Hence, $\boxed{f''(x) = 6x}$

(c) $f'''(x) = \frac{d}{dx}[f''(x)] = \frac{d}{dx}[6x] \overset{D.5}{=} 6$ Hence, $\boxed{f'''(x) = 6}$

(d) 'Slope of tangent line to curve f at $x = -2$' $\iff$ $f'(-2) = 3(-2)^2 = \boxed{12}$

EXAMPLE: Given implicit function $x^2 - y^5 = 99$, find : (a) $\frac{dy}{dx}$ (b) slope of tangent line to curve at point $(10, 1)$ (c) $\frac{d^2y}{dx^2}$

(a) $\frac{d}{dx}\left[x^2 - y^5\right] = \frac{d}{dx}[99] \overset{D.5}{\Longrightarrow} 2x - 5y^4\frac{dy}{dx} = 0 \Longrightarrow \boxed{\frac{dy}{dx} = \frac{2x}{5y^4}}$

(b) 'Slope of tangent line to curve at point $(10, 1)$' $\iff$ $\left.\frac{dy}{dx}\right|_{(x,y)=(10,1)} = \frac{2(10)}{5(1)^4} = \boxed{4}$

(c) $\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{dy}{dx}\right] = \frac{d}{dx}\left[\frac{2x}{5y^4}\right] \overset{D.4}{=} \frac{5y^4(2) - 2x(20y^3)\frac{dy}{dx}}{(5y^4)^2} = \frac{10y^4 - 40xy^3\left(\frac{2x}{5y^4}\right)}{(5y^4)^2} = \frac{50y^5 - 80x^2}{125y^9} \Longrightarrow \boxed{\frac{d^2y}{dx^2} = \frac{50y^5 - 80x^2}{125y^9}}$
