INTEGRATION BY SUBSTITUTION

EXAMPLE: Evaluate $\int (3x-6)^{12}\ dx$.

(Change of variables [CV]) Let $u = 3x - 6$. Then $\dfrac{du}{dx} = 3 \implies du = 3dx \implies dx = \dfrac{1}{3}du$.

$$\implies \int (3x-6)^{12}\ dx \overset{CV}{=} \int u^{12}\left(\frac{1}{3}du\right) \overset{INT.2}{=} \frac{1}{3}\int u^{12}\ du \overset{INT.4}{=} \frac{1}{3}\left[\frac{1}{13}u^{13}\right] + C = \frac{1}{39}u^{13} + C \overset{CV}{=} \boxed{\frac{1}{39}(3x-6)^{13} + C}$$

EXAMPLE: Evaluate $\displaystyle\int_e^{e^2} \frac{1}{x\sqrt{\ln x}}\ dx$.

(Change of variables [CV]) Let $u = \ln x$. Then $\dfrac{du}{dx} = \dfrac{1}{x} \implies du = \dfrac{1}{x}dx$.

Find the corresponding values of u at the bounds of integration: $u(e) = \ln e = 1, \quad u\left(e^2\right) = \ln e^2 = 2$.

$$\implies \int_e^{e^2} \frac{1}{x\sqrt{\ln x}}\ dx \overset{CV}{=} \int_1^2 \frac{1}{\sqrt{u}}\ du = \int_1^2 u^{-1/2}\ du \overset{INT.4}{=} \left[\frac{1}{-\frac{1}{2}+1}u^{-1/2+1}\right]_{u=1}^{u=2} = \Big[2\sqrt{u}\Big]_{u=1}^{u=2} \overset{FTC.1}{=} 2\sqrt{2} - 2\sqrt{1} = \boxed{2\sqrt{2} - 2}$$

EXAMPLE: Evaluate $\int \sqrt[3]{t^3 + t^5}\ dt$.

First, some algebraic simplification: $\int \sqrt[3]{t^3+t^5}\ dt = \int \sqrt[3]{t^3(1+t^2)}\ dt = \int \sqrt[3]{t^3}\,\sqrt[3]{1+t^2}\ dx = \int t\sqrt[3]{1+t^2}\ dt$

(Change of variables [CV]) Let $u = 1 + t^2$. Then $\dfrac{du}{dt} = 2t \implies du = 2t\ dt \implies t\ dt = \dfrac{1}{2}du$.

$$\implies \int t\sqrt[3]{t^3+t^5}\ dt \overset{CV}{=} \int \sqrt[3]{u}\left(\frac{1}{2}du\right) \overset{INT.2}{=} \frac{1}{2}\int u^{1/3}\ du \overset{INT.4}{=} \frac{1}{2}\left[\frac{1}{\frac{1}{3}+1}u^{1/3+1}\right] + C = \frac{3}{8}u^{4/3} + C \overset{CV}{=} \boxed{\frac{3}{8}(1+t^2)^{4/3} + C}$$

EXAMPLE: Evaluate $\displaystyle\int_0^1 \frac{u}{8u+2}\ du$.

(Change of variables [CV]) Let $w = 8u + 2$. Then $\dfrac{dw}{du} = 8 \implies dw = 8\ du \implies du = \dfrac{1}{8}dw$.

Now, we still need to substitute for the u on top: $w = 8u + 2 \iff 8u = w - 2 \iff u = \dfrac{1}{8}(w-2)$.

Find the corresponding values of w at the bounds of integration: $w(0) = 8(0) + 2 = 2, \quad w(1) = 8(1) + 2 = 10$.

$$\implies \int_0^1 \frac{u}{8u+2}\ du \overset{CV}{=} \int_2^{10} \left(\frac{1}{8}(w-2)\right)\left(\frac{1}{w}\right)\left(\frac{1}{8}dw\right) \overset{DINT.3}{=} \frac{1}{64}\int_2^{10} \frac{w-2}{w}\ dw = \frac{1}{64}\int_2^{10}\left(1 - \frac{2}{w}\right)\ dw$$

$$= \frac{1}{64}\Big[w - 2\ln|w|\Big]_{w=2}^{w=10} \overset{FTC.1}{=} \frac{1}{64}\Big[(10 - 2\ln|10|) - (2 - 2\ln|2|)\Big] = \boxed{\frac{1}{32}\Big[4 - \ln 5\Big] \approx 0.07471}$$

EXAMPLE: Evaluate $\displaystyle\int_1^{10} (1-y)^7\ dy$.

(Change of variables [CV]) Let $u = 1 - y$. Then $\dfrac{du}{dy} = -1 \implies du = -dy \implies dy = -du$.

Find the corresponding values of u at the bounds of integration: $u(1) = 1 - (1) = 0, \quad u(10) = 1 - (10) = -9$.

IMPORTANT: Even though $-9 < 0$, upon substitution -9 is still the upper bound of integration.

$$\implies \int_1^{10} (1-y)^7\ dy \overset{CV}{=} \int_0^{-9} u^7(-du) \overset{DINT.3}{=} -\int_0^{-9} u^7\ du \overset{DINT.1}{=} \int_{-9}^0 u^7\ du \overset{INT.4}{=} \left[\frac{1}{7+1}u^{7+1}\right]_{u=-9}^{u=0}$$

$$= \left[\frac{1}{8}u^8\right]_{u=-9}^{u=0} \overset{FTC.1}{=} \frac{1}{8}\left[(0)^8 - (-9)^8\right] = \boxed{-\frac{43046721}{8} = -5380840.125}$$