LIMITS OF FUNCTIONS (I)

ELEMENTARY FUNCTIONS: (Elementary functions are NOT piecewise functions)

- Constant Functions: $f(x) = k$, where $k \in \mathbb{R}$
- Polynomials: $f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n$, where $a_0, a_1, \ldots, a_n \in \mathbb{R}$
- Rational Functions: $f(x) = \dfrac{P(x)}{Q(x)}$, where P, Q are polynomials.
- Exponentials: $f \in \{e^x, b^x\}$, where $b \in \mathbb{R}$
- Logarithms: $f \in \{\ln x, \log x, \log_b x\}$, where $b \in \mathbb{R}$
- Trig Functions: $f \in \{\sin x, \cos x, \tan x, \csc x, \sec x, \cot x\}$
- Inverse Trig: $f \in \{\arcsin x, \arccos x, \arctan x, \text{arccsc } x, \text{arcsec } x, \text{arccot } x\}$
- Any **finite** sum, difference, product, quotient, power, root, or composition of any of the above elementary functions.

PIECEWISE FUNCTIONS:

- A **piecewise function** has more than one function (piece) in its definition, spanned by a single large left brace.
- e.g. $f(x) = \begin{cases} 2x-1 & \text{, if } x < 2 \\ 1+\sqrt{x} & \text{, if } x \geq 2 \end{cases} = \begin{cases} 2x-1 & \text{, if } x \in (-\infty, 2) \\ 1+\sqrt{x} & \text{, if } x \in [2, \infty) \end{cases} = \begin{cases} 2x-1 & \text{, if } x < 2 \\ 1+\sqrt{x} & \text{, otherwise} \end{cases}$

  $\implies f(25) = 1 + \sqrt{25} = 6, \qquad f(2) = 1 + \sqrt{2}, \qquad f(-3) = 2(-3) - 1 = -7$
- e.g. $|x| = \begin{cases} x & \text{, if } x \geq 0 \\ -x & \text{, if } x < 0 \end{cases} = \begin{cases} x & \text{, if } x \in [0, \infty) \\ -x & \text{, if } x \in (-\infty, 0) \end{cases} = \begin{cases} x & \text{, if } x \in \mathbb{R}_+ \cup \{0\} \\ -x & \text{, if } x \in \mathbb{R}_- \end{cases} = \begin{cases} x & \text{, if } x \geq 0 \\ -x & \text{, otherwise} \end{cases}$

  $\implies |10| = 10, \qquad |0| = 0, \qquad |-7| = -(-7) = 7$
- e.g. $g(t) = \begin{cases} 1+t^2 & \text{, if } t < -\pi/2 \\ \sin t & \text{if } -\pi/2 \leq t < \pi \\ 20 & \text{, if } t = \pi \\ \log t & \text{if } t > \pi \end{cases} = \begin{cases} 1+t^2 & \text{, if } t \in (-\infty, -\pi/2) \\ \sin t & \text{if } t \in [-\pi/2, \pi) \\ 20 & \text{, if } t \in \{\pi\} \\ \log t & \text{if } t \in (\pi, \infty) \end{cases} = \begin{cases} 1+t^2 & \text{, if } t < -\pi/2 \\ \sin t & \text{if } -\pi/2 \leq t < \pi \\ 20 & \text{, if } t = \pi \\ \log t & \text{otherwise} \end{cases}$

  $\implies g(-8) = 1 + (-8)^2 = 65, \quad g\left(-\frac{\pi}{4}\right) = \sin\left(-\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}, \quad g(\pi) = 20, \quad g(1000) = \log 1000 = \log 10^3 = 3$

INTERESTING PROPERTIES OF INFINITY:

- Remember, ∞ is **not a real number**, but rather a **symbol** indicating **growth without bound**.
- Similarly, $-\infty$ indicates **decay without bound**.
- However, even though $\pm\infty$ are symbols, they satisfy some arithmetic properties that agree with intuition:
- (E.1) $\infty + \infty = \infty \qquad -\infty - \infty = -\infty$
- (E.2) $\forall x \in \mathbb{R}, \quad \infty + x = x + \infty = \infty \quad \text{and} \quad -\infty + x = x - \infty = -\infty$
- (E.3) $(\infty)(\infty) = \infty, \qquad (-\infty)(-\infty) = \infty, \qquad (-\infty)(\infty) = (\infty)(-\infty) = -\infty$
- (E.4) $x > 0 \implies x \cdot \infty = \infty$ and $x \cdot (-\infty) = -\infty, \qquad x < 0 \implies x \cdot \infty = -\infty$ and $x \cdot (-\infty) = \infty$
- (E.5) $n \in \mathbb{N} \implies \infty^n = \infty \quad \text{and} \quad \sqrt[n]{\infty} = \infty$

INDETERMINANT FORMS: $\dfrac{0}{0}, \dfrac{\infty}{\infty}, 0 \cdot \infty, \infty - \infty, 0^0, \infty^0, 1^\infty$

At this point (i.e. early in the course), only the indeterminant forms $\dfrac{0}{0}$ and $\dfrac{\infty}{\infty}$ will be encountered.

LIMIT DEFINITIONS/NOTATION:

- 'The **value** of $f(x)$ **at** $x = c$ is y' $\iff$ $f(c) = y$
- (2-sided limit) 'The **limit** of $f(x)$ as x **approaches** c is L' $\iff$ $\lim_{x\to c} f(x) = L$ $\iff$ $f(x) \to L$ as $x \to c$
- (1-sided limit) 'The **limit** of $f(x)$ as x **approaches** c **from the left (negative) side** is L' $\iff$ $\lim_{x\to c^-} f(x) = L$
- (1-sided limit) 'The **limit** of $f(x)$ as x **approaches** c **from the right (positive) side** is L' $\iff$ $\lim_{x\to c^+} f(x) = L$
- (infinite limit) 'f **increases w/o bound** as x approaches c' $\iff$ 'f **blows up** as x approaches c' $\iff$ $\lim_{x\to c} f(x) = +\infty$
- (infinite limit) 'f **decreases w/o bound** as x approaches c' $\iff$ 'f **nose-dives** as x approaches c' $\iff$ $\lim_{x\to c} f(x) = -\infty$
- REMARK: When x approaches c, it only makes sense that x be **sufficiently close** to c.

RELATIONSHIP BETWEEN 2-SIDED & 1-SIDED LIMITS: $\lim_{x\to c} f(x) = L \iff \lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x) = L$

LIMIT RULES: DNE means '**D**oes **N**ot **E**xist'

- (L.0) (Constant Rule) $\lim_{x\to c} k = k$, where $k \in \mathbb{R}$
- (L.1) (Multiple Rule) $\lim_{x\to c} [kf(x)] = k \lim_{x\to c} f(x)$, where $k \in \mathbb{R}$
- (L.2) (Sum/Diff Rule) $\lim_{x\to c} [f(x) \pm g(x)] = \lim_{x\to c} f(x) \pm \lim_{x\to c} g(x)$
- (L.3) (Product Rule) $\lim_{x\to c} [f(x)g(x)] = \left[\lim_{x\to c} f(x)\right] \left[\lim_{x\to c} g(x)\right]$
- (L.4) (Quotient Rule) $\lim_{x\to c} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}$, provided $\lim_{x\to c} g(x) \neq 0$
- (L.5) (Power Rule) $\lim_{x\to c} [f(x)]^n = \left[\lim_{x\to c} f(x)\right]^n$, where $n \in \mathbb{Q}$ and $\lim_{x\to c} f(x)$ exists
- (L.6) (DNE Rule) After simplification, limit of part of a function is DNE $\implies$ limit of entire function is DNE.

SPECIAL LIMITS:

- (S.1) $\lim_{x\to 0^+} \dfrac{1}{x^n} = \lim_{x\to 0^+} \dfrac{1}{\sqrt[n]{x}} = +\infty, \ \forall n \in \mathbb{N}$ $\quad \lim_{x\to 0^-} \dfrac{1}{x^n} = \begin{cases} +\infty & \text{, if } n \in \mathbb{N} \text{ is even} \\ -\infty & \text{, if } n \in \mathbb{N} \text{ is odd} \end{cases}$ $\quad \lim_{x\to 0^-} \dfrac{1}{\sqrt[n]{x}} = -\infty, \ \forall \text{ odd } n \in \mathbb{N}$
- (S.2) $\lim_{x\to 0} \dfrac{1}{x^n} = +\infty \ \ \forall \text{ even } n \in \mathbb{N}$ $\quad \lim_{x\to 0} \dfrac{1}{x^n} = \lim_{x\to 0} \dfrac{1}{\sqrt[n]{x}} = \text{DNE} \ \ \forall \text{ odd } n \in \mathbb{N}$
- (S.3) $\lim_{x\to 0} \dfrac{\sin x}{x} = 1$ $\quad \lim_{x\to 0} \dfrac{\cos x - 1}{x} = 0$ $\quad \lim_{x\to 0^+} \ln x = \lim_{x\to 0^+} \log x = \lim_{x\to 0^+} \log_b x = -\infty$
- Let $\mathcal{A} \in \left\{\cdots, -\frac{5\pi}{2}, -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \cdots\right\}$. Then: $\lim_{x\to \mathcal{A}^-} \tan x = +\infty$ $\quad \lim_{x\to \mathcal{A}^+} \tan x = -\infty$ $\quad \lim_{x\to \mathcal{A}} \sec x = \text{DNE}$
- Let $\mathcal{B} \in \{\ldots, -3\pi, -2\pi, -\pi, 0, \pi, 2\pi, 3\pi, \ldots\}$. Then: $\lim_{x\to \mathcal{B}^-} \cot x = -\infty$ $\quad \lim_{x\to \mathcal{B}^+} \cot x = +\infty$ $\quad \lim_{x\to \mathcal{B}} \csc x = \text{DNE}$

ALGEBRAIC PROCEDURE FOR LIMITS OF ELEMENTARY FUNCTIONS:

- First, compute the **2-sided limit**: $\lim_{x\to c} f(x)$
- If $\lim_{x\to c} f(x) = L$, then $\lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x) = L$. Else, $\lim_{x\to c} f(x) = \text{DNE} \implies$ 1-sided limits must be formally computed.
- (NS) (**naïve substitution**) If $f(c)$ is defined, that is, $f(c) = L \in \mathbb{R}$, then $\lim_{x\to c} f(x) = \lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x) = L$
- (**indeterminant**) If $f(c) = \dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$, rewrite and/or simplify $f(x)$ first, then apply naïve substitution to get the answer.
  Simplifications: Rationalize numerator/denominator, combine fractions, factor polynomial(s), mold into special limit, ...
- (CV) (**transformation**) Sometimes to **mold** a limit into one of the special limits, a **change of variables** is necessary.
- If $f(c) = \dfrac{a}{0}$, $a \neq 0$, then use **limit rules**, **simplification**, and/or **transformation** to reduce to a limit of $\dfrac{1}{x^n}$ or $\dfrac{1}{\sqrt[n]{x}}$
- (**trig limits**) For limits of trig, sometimes it's best to rewrite trig expression in terms of $\sin(\cdot)$ and $\cos(\cdot)$ first.

ALGEBRAIC PROCEDURE FOR LIMITS OF PIECEWISE FUNCTIONS:

- First, compute the **1-sided limits**: $\lim_{x\to c^-} f(x)$ and $\lim_{x\to c^+} f(x)$
- If $\lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x) = L$, then the **2-sided limit** $\lim_{x\to c} f(x) = L$. Else, $\lim_{x\to c} f(x) = \text{DNE}$

EXAMPLE: Evaluate: $\lim_{x\to -2}(3x^3-x-1)$, $\lim_{x\to(-2)^-}(3x^3-x-1)$, and $\lim_{x\to(-2)^+}(3x^3-x-1)$

Since the function is **elementary**, find the **2-sided limit** first.

Try naïve substitution: $\lim_{x\to -2}(3x^3-x-1) \overset{NS}{=} 3(-2)^3-(-2)-1=-23$

Thus, $\boxed{\lim_{x\to -2}(3x^3-x-1)=-23} \implies \boxed{\lim_{x\to(-2)^-}(3x^3-x-1)=\lim_{x\to(-2)^+}(3x^3-x-1)=-23}$

EXAMPLE: Evaluate $\lim_{w\to\sqrt{3}}\frac{w}{w^2-1}$

Try naïve substitution: $\lim_{w\to\sqrt{3}}\frac{w}{w^2-1} \overset{NS}{=} \frac{(\sqrt{3})}{(\sqrt{3})^2-1}=\frac{\sqrt{3}}{2}$. Thus, $\boxed{\lim_{w\to\sqrt{3}}\frac{w}{w^2-1}=\frac{\sqrt{3}}{2}}$

EXAMPLE: Evaluate $\lim_{\theta\to 7\pi/6}\frac{\sin^2\theta}{\theta}$

Try naïve substitution: $\lim_{\theta\to 7\pi/6}\frac{\sin^2\theta}{\theta} \overset{NS}{=} \frac{\sin^2\left(\frac{7\pi}{6}\right)}{\left(\frac{7\pi}{6}\right)}=\frac{\frac{1}{4}}{\frac{7\pi}{6}}=\frac{1}{4}\div\frac{7\pi}{6}=\frac{1}{4}\cdot\frac{6}{7\pi}=\frac{3}{14\pi}$. Thus, $\boxed{\lim_{\theta\to 7\pi/6}\frac{\sin^2\theta}{\theta}=\frac{3}{14\pi}}$

EXAMPLE: Evaluate $\lim_{Y\to 1/3}\frac{21Y^3-7Y^2+3Y-1}{15Y^2+16Y-7}$

Try naïve substitution: $\lim_{Y\to 1/3}\frac{21Y^3-7Y^2+3Y-1}{15Y^2+16Y-7} \overset{NS}{=} \frac{21\left(\frac{1}{3}\right)^3-7\left(\frac{1}{3}\right)^2+3\left(\frac{1}{3}\right)-1}{15\left(\frac{1}{3}\right)^2+16\left(\frac{1}{3}\right)-7}=\frac{0}{0} \implies$ rewrite/simplify function!!

Here, **factor** numerator & denominator: $\frac{21Y^3-7Y^2+3Y-1}{15Y^2+16Y-7}=\frac{7Y^2(3Y-1)+(3Y-1)}{(5Y+7)(3Y-1)}=\frac{(7Y^2+1)(3Y-1)}{(5Y+7)(3Y-1)}$

Now, simplify & try naïve substitution again:

$$\lim_{Y\to 1/3}\frac{21Y^3-7Y^2+3Y-1}{15Y^2+16Y-7}=\lim_{Y\to 1/3}\frac{(7Y^2+1)(3Y-1)}{(5Y+7)(3Y-1)}=\lim_{Y\to 1/3}\frac{7Y^2+1}{5Y+7} \overset{NS}{=} \frac{7\left(\frac{1}{3}\right)^2+1}{5\left(\frac{1}{3}\right)+7}=\frac{\frac{16}{9}}{\frac{26}{3}}=\frac{16}{9}\cdot\frac{3}{26}=\frac{8}{39}$$

Thus, $\boxed{\lim_{Y\to 1/3}\frac{21Y^3-7Y^2+3Y-1}{15Y^2+16Y-7}=\frac{8}{39}}$

EXAMPLE: Evaluate $\lim_{\alpha\to(-1)^+}\frac{1+\frac{1}{\alpha}}{\alpha^2-1}$

Since the function is **elementary**, find the **2-sided limit** first.

Try naïve substitution: $\lim_{\alpha\to -1}\frac{1+\frac{1}{\alpha}}{\alpha^2-1} \overset{NS}{=} \frac{1+\frac{1}{(-1)}}{(-1)^2-1}=\frac{0}{0} \implies$ rewrite/simplify function!!

Here, **combine terms** of numerator into one fraction & **factor** denominator: $\frac{1+\frac{1}{\alpha}}{\alpha^2-1}=\frac{\frac{\alpha}{\alpha}+\frac{1}{\alpha}}{\alpha^2-1}=\frac{\frac{\alpha+1}{\alpha}}{(\alpha+1)(\alpha-1)}$

Now, simplify & try naïve substitution again:

$$\lim_{\alpha\to -1}\frac{1+\frac{1}{\alpha}}{\alpha^2-1}=\lim_{\alpha\to -1}\frac{\frac{\alpha+1}{\alpha}}{(\alpha+1)(\alpha-1)}=\lim_{\alpha\to -1}\frac{\alpha+1}{\alpha}\cdot\frac{1}{(\alpha+1)(\alpha-1)}=\lim_{\alpha\to -1}\frac{1}{\alpha(\alpha-1)} \overset{NS}{=} \frac{1}{(-1)\left[(-1)-1\right]}=\frac{1}{2}$$

Thus, $\lim_{\alpha\to -1}\frac{1+\frac{1}{\alpha}}{\alpha^2-1}=\frac{1}{2} \implies \lim_{\alpha\to(-1)^-}\frac{1+\frac{1}{\alpha}}{\alpha^2-1}=\boxed{\lim_{\alpha\to(-1)^+}\frac{1+\frac{1}{\alpha}}{\alpha^2-1}=\frac{1}{2}}$

EXAMPLE: Evaluate $\lim_{t\to 16}\frac{\sqrt{t}-4}{t-16}$

Try naïve substitution: $\lim_{t\to 16}\frac{\sqrt{t}-4}{t-16} \overset{NS}{=} \frac{\sqrt{(16)}-4}{(16)-16}=\frac{0}{0} \implies$ rewrite/simplify function!!

Here, **rationalize** numerator: $\frac{\sqrt{t}-4}{t-16}=\frac{-4+\sqrt{t}}{t-16}\cdot\frac{-4-\sqrt{t}}{-4-\sqrt{t}}=\frac{(-4)^2-(\sqrt{t})^2}{(t-16)(-4-\sqrt{t})}=\frac{16-t}{(t-16)(-4-\sqrt{t})}$

Now, simplify & try naïve substitution again:

$$\lim_{t\to 16}\frac{\sqrt{t}-4}{t-16}=\lim_{t\to 16}\frac{16-t}{(t-16)(-4-\sqrt{t})}=\lim_{t\to 16}\frac{1}{4+\sqrt{t}} \overset{NS}{=} \frac{1}{4+\sqrt{(16)}}=\frac{1}{8}$$

Thus, $\boxed{\lim_{t\to 16}\frac{\sqrt{t}-4}{t-16}=\frac{1}{8}}$

EXAMPLE: Evaluate: $\lim\limits_{\theta\to 0} \csc\theta$, $\lim\limits_{\theta\to 0^-} \csc\theta$, and $\lim\limits_{\theta\to 0^+} \csc\theta$

Since the function is **elementary**, find the **2-sided limit** first.

First, write function in terms of $\sin\theta, \cos\theta$: $\csc\theta = \dfrac{1}{\sin\theta}$

Try naïve substitution: $\lim\limits_{\theta\to 0} \csc\theta = \lim\limits_{\theta\to 0} \dfrac{1}{\sin\theta} \overset{NS}{=} \dfrac{1}{\sin 0} = \dfrac{1}{0} \implies$ reduce limit to one involving $\dfrac{1}{\theta^n}$ or $\dfrac{1}{\sqrt[n]{\theta}}$, where $n \in \mathbb{N}$

Mold function into one involving $\dfrac{\sin\theta}{\theta}$: $\csc\theta = \dfrac{1}{\sin\theta} = \dfrac{1}{\sin\theta}\cdot\dfrac{\theta}{\theta} = \dfrac{\theta}{\sin\theta}\cdot\dfrac{1}{\theta} = \dfrac{1}{\frac{\sin\theta}{\theta}}\cdot\dfrac{1}{\theta}$

$$\implies \lim_{\theta\to 0}\csc\theta = \lim_{\theta\to 0}\frac{1}{\frac{\sin\theta}{\theta}}\cdot\frac{1}{\theta} \overset{L.3}{=} \left(\lim_{\theta\to 0}\frac{1}{\frac{\sin\theta}{\theta}}\right)\left(\lim_{\theta\to 0}\frac{1}{\theta}\right) \overset{L.4}{=} \left(\frac{1}{\lim\limits_{\theta\to 0}\frac{\sin\theta}{\theta}}\right)\left(\lim_{\theta\to 0}\frac{1}{\theta}\right) \overset{S.3}{=} \left[\frac{1}{(1)}\right]\left(\lim_{\theta\to 0}\frac{1}{\theta}\right) = \lim_{\theta\to 0}\frac{1}{\theta} \overset{S.2}{=} \boxed{\text{DNE}}$$

Now, since the 2-sided limit **DNE**, find the 1-sided limits **formally** (recycling work already performed):

$$\implies \lim_{\theta\to 0^-}\csc\theta = \left(\frac{1}{\lim\limits_{\theta\to 0^-}\frac{\sin\theta}{\theta}}\right)\left(\lim_{\theta\to 0^-}\frac{1}{\theta}\right) \overset{S.3}{=} \left[\frac{1}{(1)}\right]\left(\lim_{\theta\to 0^-}\frac{1}{\theta}\right) = \lim_{\theta\to 0^-}\frac{1}{\theta} \overset{S.1}{=} \boxed{-\infty}$$

$$\implies \lim_{\theta\to 0^+}\csc\theta = \left(\frac{1}{\lim\limits_{\theta\to 0^+}\frac{\sin\theta}{\theta}}\right)\left(\lim_{\theta\to 0^+}\frac{1}{\theta}\right) \overset{S.3}{=} \left[\frac{1}{(1)}\right]\left(\lim_{\theta\to 0^+}\frac{1}{\theta}\right) = \lim_{\theta\to 0^+}\frac{1}{\theta} \overset{S.1}{=} \boxed{+\infty}$$

EXAMPLE: Evaluate: $\lim\limits_{x\to 3}\dfrac{1}{2x^2+4x-30}$, $\lim\limits_{x\to 3^-}\dfrac{1}{2x^2+4x-30}$, and $\lim\limits_{x\to 3^+}\dfrac{1}{2x^2+4x-30}$

Since the function is **elementary**, find the **2-sided limit** first.

Try naïve substitution: $\lim\limits_{x\to 3}\dfrac{1}{2x^2+4x-30} \overset{NS}{=} \dfrac{1}{2(3)^2+4(3)-30} = \dfrac{1}{0} \implies$ reduce limit to $\lim\limits_{x\to 0}\dfrac{1}{x^n}$, where $n\in\mathbb{N}$

Factor function: $\dfrac{1}{2x^2+4x-30} = \dfrac{1}{2(x^2+2x-15)} = \dfrac{1}{2(x-3)(x+5)} = \left(\dfrac{1}{2}\right)\left(\dfrac{1}{x-3}\right)\left(\dfrac{1}{x+5}\right)$

$$\implies \lim_{x\to 3}\frac{1}{2x^2+4x-30} = \lim_{x\to 3}\left(\frac{1}{2}\right)\left(\frac{1}{x-3}\right)\left(\frac{1}{x+5}\right) \overset{L.3}{=} \left(\lim_{x\to 3}\frac{1}{2}\right)\left(\lim_{x\to 3}\frac{1}{x-3}\right)\left(\lim_{x\to 3}\frac{1}{x+5}\right) \overset{L.0}{=} \left(\frac{1}{2}\right)\left(\lim_{x\to 3}\frac{1}{x+5}\right)\left(\lim_{x\to 3}\frac{1}{x-3}\right)$$

$\overset{NS}{=} \left(\dfrac{1}{2}\right)\left(\dfrac{1}{(3)+5}\right)\left(\lim\limits_{x\to 3}\dfrac{1}{x-3}\right) = \dfrac{1}{16}\lim\limits_{x\to 3}\dfrac{1}{x-3}$, which is very close in form to $\lim\limits_{x\to 0}\dfrac{1}{x}$, but not quite!

Hence, use a **transformation** (AKA change of variables):

Let $u = x-3$. Then, $x = u+3$, which means $x\to 3 \iff (u+3)\to 3 \iff u\to 0$

So, $\lim\limits_{x\to 3}\dfrac{1}{2x^2+4x-30} = \dfrac{1}{16}\lim\limits_{x\to 3}\dfrac{1}{x-3} \overset{CV}{=} \dfrac{1}{16}\lim\limits_{u\to 0}\dfrac{1}{u} \overset{S.2}{=} \dfrac{1}{16}(DNE) \overset{L.6}{=} \boxed{DNE}$

Now, since the 2-sided limit **DNE**, find the 1-sided limits **formally** (recycling work already performed):

$$\implies \lim_{x\to 3^-}\frac{1}{2x^2+4x-30} = \frac{1}{16}\lim_{x\to 3^-}\frac{1}{x-3} \overset{CV}{=} \frac{1}{16}\lim_{u\to 0^-}\frac{1}{u} \overset{S.1}{=} \frac{1}{16}(-\infty) \overset{E.4}{=} \boxed{-\infty}$$

$$\implies \lim_{x\to 3^+}\frac{1}{2x^2+4x-30} = \frac{1}{16}\lim_{x\to 3^+}\frac{1}{x-3} \overset{CV}{=} \frac{1}{16}\lim_{u\to 0^+}\frac{1}{u} \overset{S.1}{=} \frac{1}{16}(+\infty) \overset{E.4}{=} \boxed{+\infty}$$

EXAMPLE: Evaluate $\lim\limits_{t\to -\pi}\dfrac{-2\sqrt{3}}{\sqrt[3]{t+\pi}}$, $\lim\limits_{t\to(-\pi)^-}\dfrac{-2\sqrt{3}}{\sqrt[3]{t+\pi}}$, and $\lim\limits_{t\to(-\pi)^+}\dfrac{-2\sqrt{3}}{\sqrt[3]{t+\pi}}$

Since the function is **elementary**, find the **2-sided limit** first.

Try naïve substitution: $\lim\limits_{t\to -\pi}\dfrac{-2\sqrt{3}}{\sqrt[3]{t+\pi}} \overset{NS}{=} \dfrac{-2\sqrt{3}}{\sqrt[3]{(-\pi)+\pi}} = \dfrac{-2\sqrt{3}}{0} \implies$ reduce limit to $\lim\limits_{t\to 0}\dfrac{1}{\sqrt[n]{t}}$, where $n\in\mathbb{N}$

$\lim\limits_{t\to -\pi}\dfrac{-2\sqrt{3}}{\sqrt[3]{t+\pi}} = -2\sqrt{3}\lim\limits_{t\to -\pi}\dfrac{1}{\sqrt[3]{t+\pi}}$, which is very close in form to $\lim\limits_{t\to 0}\dfrac{1}{\sqrt[3]{t}}$, but not quite!

Hence, use a **transformation** (AKA change of variables):

Let $u = t+\pi$. Then, $t = u-\pi$, which means $t\to -\pi \iff (u-\pi)\to -\pi \iff u\to 0$

So, $\lim\limits_{t\to -\pi}\dfrac{-2\sqrt{3}}{\sqrt[3]{t+\pi}} \overset{CV}{=} -2\sqrt{3}\lim\limits_{u\to 0}\dfrac{1}{\sqrt[3]{u}} \overset{S.2}{=} -2\sqrt{3}(DNE) \overset{L.6}{=} \boxed{DNE}$

Now, since the 2-sided limit **DNE**, find the 1-sided limits **formally** (recycling work already performed):

$$\lim_{t\to(-\pi)^-}\frac{-2\sqrt{3}}{\sqrt[3]{t+\pi}} \overset{CV}{=} -2\sqrt{3}\lim_{u\to 0^-}\frac{1}{\sqrt[3]{u}} \overset{S.1}{=} -2\sqrt{3}(-\infty) \overset{E.4}{=} \boxed{+\infty}$$

$$\lim_{t\to(-\pi)^+}\frac{-2\sqrt{3}}{\sqrt[3]{t+\pi}} \overset{CV}{=} -2\sqrt{3}\lim_{u\to 0^+}\frac{1}{\sqrt[3]{u}} \overset{S.1}{=} -2\sqrt{3}(+\infty) \overset{E.4}{=} \boxed{-\infty}$$

EXAMPLE: Evaluate: $\lim\limits_{Z\to\sqrt{7}} \dfrac{1}{(Z-\sqrt{7})^2}$, $\lim\limits_{Z\to\sqrt{7}^-} \dfrac{1}{(Z-\sqrt{7})^2}$, and $\lim\limits_{Z\to\sqrt{7}^+} \dfrac{1}{(Z-\sqrt{7})^2}$

Since the function is **elementary**, find the **2-sided limit** first.

Try naïve substitution: $\lim\limits_{Z\to\sqrt{7}} \dfrac{1}{(Z-\sqrt{7})^2} \overset{NS}{=} \dfrac{1}{0} \implies$ reduce limit to $\lim\limits_{Z\to 0} \dfrac{1}{Z^n}$, where $n \in \mathbb{N}$

The limit is very close in form to $\lim\limits_{Z\to 0} \dfrac{1}{Z^2}$, but not quite!

Hence, use a **transformation** (AKA change of variables):

Let $u = Z - \sqrt{7}$. Then, $Z = u + \sqrt{7}$, which means $Z \to \sqrt{7} \iff (u+\sqrt{7}) \to \sqrt{7} \iff u \to 0$

So, $\lim\limits_{Z\to\sqrt{7}} \dfrac{1}{(Z-\sqrt{7})^2} \overset{CV}{=} \lim\limits_{u\to 0} \dfrac{1}{u^2} \overset{S.2}{=} \boxed{+\infty}$

Now, since the 2-sided limit exists, the 1-sided limits share the same value as the 2-sided limit:

$$\lim_{Z\to\sqrt{7}^-} \frac{1}{(Z-\sqrt{7})^2} = \lim_{Z\to\sqrt{7}^+} \frac{1}{(Z-\sqrt{7})^2} = \boxed{+\infty}$$

EXAMPLE: Let $f(x) = \begin{cases} \dfrac{1}{x^4} & , \text{if } x < \pi/3 \\ \sin x & , \text{if } x \geq \pi/3 \end{cases}$ Evaluate: (a) $f(\pi/3)$ (b) $\lim\limits_{x\to\pi/3} f(x)$

(a) $f\left(\frac{\pi}{3}\right) = \sin\left(\frac{\pi}{3}\right) = \boxed{\dfrac{\sqrt{3}}{2}}$

(b) Since f is **piecewise**, find the **1-sided limits** first.

$$\lim_{x\to\pi/3^-} f(x) = \frac{1}{\left(\frac{\pi}{3}\right)^4} = \frac{1}{\left(\frac{\pi^4}{81}\right)} = \frac{81}{\pi^4} \qquad \lim_{x\to\pi/3^+} f(x) = \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$$

Thus, since $\lim\limits_{x\to\pi/3^-} f(x) \neq \lim\limits_{x\to\pi/3^+} f(x)$, it follows that $\lim\limits_{x\to\pi/3} f(x) = \boxed{DNE}$

EXAMPLE: Let $g(t) = \begin{cases} t-3 & , \text{if } t < -1 \\ 2t-1 & , \text{if } -1 \leq t < 1 \\ 25 & , \text{if } t = 1 \\ \sqrt{t} & , \text{if } 1 < t \leq 4 \\ t^2+t & , \text{if } t > 4 \end{cases}$ Evaluate: (a) $g(1)$ (b) $\lim\limits_{t\to 1} g(t)$

(a) $g(1) = \boxed{25}$

(b) Since g is **piecewise**, find the **1-sided limits** first.

$$\lim_{t\to 1^-} g(t) = 2(1) - 1 = 1 \qquad \lim_{t\to 1^+} g(t) = \sqrt{(1)} = 1$$

Thus, since $\lim\limits_{t\to 1^-} g(t) = \lim\limits_{t\to 1^+} g(t)$, it follows that $\lim\limits_{t\to 1} g(t) = \boxed{1}$