RELATED RATES

EXAMPLE: Suppose the length, ℓ, of a rectangle is increasing at 2 in/sec and the width, w, decreases at a rate of 3 in/sec.

Moreover, suppose at the beginning (time $t = 0$ sec) the width is 30 in and the length is 10 in.

a) How fast are the perimeter & area changing when the width is 15 in and the length is 20 in?

STEP 1: Interpret all the given information in the problem:

"length is 20 in" $\iff \ell = 20$ in   "width is 15 in" $\iff w = 15$ in

"length is increasing at 2 in/sec" $\iff \frac{d\ell}{dt} = 2$ in/sec   "width decreases at a rate of 3 in/sec" $\iff \frac{dw}{dt} = -3$ in/sec

"How fast is the perimeter changing?" $\iff$ Find $\frac{dP}{dt}$ "How fast is the area changing?" $\iff$ Find $\frac{dA}{dt}$

STEP 2: Recognize the geometric or physical relationship among the variables A, P, ℓ, w (Draw a picture if necessary):

The area & perimeter formulas for a rectangle are as follow, so no picture is necessary: $A = \ell w$, $P = 2\ell + 2w$

STEP 3: Implicitly differentiate the area & perimeter formulas with respect to time (t):

$$\frac{dA}{dt} = \frac{d}{dt}[\ell w] \overset{D.3}{=} \ell\frac{dw}{dt} + w\frac{d\ell}{dt}$$

$$\frac{dP}{dt} = \frac{d}{dt}[2\ell + 2w] \overset{D.2}{=} \frac{d}{dt}[2\ell] + \frac{d}{dt}[2w] \overset{D.1}{=} 2\frac{d\ell}{dt} + 2\frac{dw}{dt}$$

STEP 4: Plug in what you know and solve for what you don't know:

$$\frac{dA}{dt} = \ell\frac{dw}{dt} + w\frac{d\ell}{dt} \implies \frac{dA}{dt} = (20 \text{ in})(-3 \text{ in/sec}) + (15 \text{ in})(2 \text{ in/sec}) \implies \frac{dA}{dt} = -30 \text{ in}^2/\text{sec}$$

$$\frac{dP}{dt} = 2\frac{d\ell}{dt} + 2\frac{dw}{dt} \implies \frac{dP}{dt} = 2(2 \text{ in/sec}) + 2(-3 \text{ in/sec}) \implies \frac{dP}{dt} = -2 \text{ in/sec}$$

STEP 5: Interpret the answers:

$\frac{dA}{dt} = -30 \text{ in}^2/\text{sec} \iff$ The area is **decreasing** at a rate of 30 in^2/sec

$\frac{dP}{dt} = -2 \text{ in/sec} \iff$ The perimeter is **decreasing** at a rate of 2 in/sec

b) How fast are the perimeter & area changing at time $t = 3$ seconds?

We aren't explicitly given the width & length at time $t = 3$ seconds, so we must determine them:

$$(\text{length at } t = 3 \text{ sec}) = (\text{length at } t = 0 \text{ sec}) + \left[\frac{d\ell}{dt} \times (3 \text{ sec})\right] = 10 \text{ in} + \left[2 \text{ in/sec} \times 3 \text{ sec}\right] = 10 \text{ in} + 6 \text{ in} = 16 \text{ in}$$

$$(\text{width at } t = 3 \text{ sec}) = (\text{width at } t = 0 \text{ sec}) + \left[\frac{dw}{dt} \times (3 \text{ sec})\right] = 30 \text{ in} + \left[-3 \text{ in/sec} \times 3 \text{ sec}\right] = 30 \text{ in} - 9 \text{ in} = 21 \text{ in}$$

The remaining steps are the same as the previous part, some of steps can be recycled:

$$\frac{dA}{dt} = \ell\frac{dw}{dt} + w\frac{d\ell}{dt} \implies \frac{dA}{dt} = (16 \text{ in})(-3 \text{ in/sec}) + (21 \text{ in})(2 \text{ in/sec}) \implies \frac{dA}{dt} = -6 \text{ in}^2/\text{sec}$$

$$\frac{dP}{dt} = 2\frac{d\ell}{dt} + 2\frac{dw}{dt} \implies \frac{dP}{dt} = 2(2 \text{ in/sec}) + 2(-3 \text{ in/sec}) \implies \frac{dP}{dt} = -2 \text{ in/sec}$$

$\frac{dA}{dt} = -6 \text{ in}^2/\text{sec} \iff$ The area is **decreasing** at a rate of 6 in^2/sec

$\frac{dP}{dt} = -2 \text{ in/sec} \iff$ The perimeter is **decreasing** at a rate of 2 in/sec

c) How fast are the perimeter & area changing the instant right before the rectangle collapses into a line?

So realize that since the **width is decreasing**, the rectangle collapses into a line the moment in time when $w = 0$.

But we need to know the **time** (t) that this occurs in order to find the corresponding length (ℓ):

$$(\text{zero width at } t = ?) = (\text{width at } t = 0) + \left[\frac{dw}{dt} \times (\text{elapsed time})\right] \implies 0 = 30 - 3t \implies w = 0 \text{ in when } t = 10 \text{ sec}$$

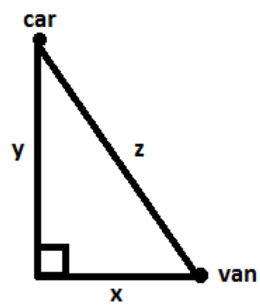
$$\implies (\text{length at } t = 10 \text{ sec}) = (\text{length at } t = 0 \text{ sec}) + \left[\frac{d\ell}{dt} \times (10 \text{ sec})\right] = 10 \text{ in} + \left[2 \text{ in/sec} \times 10 \text{ sec}\right] = 10 \text{ in} + 20 \text{ in} = 30 \text{ in}$$

$\implies \frac{dA}{dt} = -90 \text{ in}^2/\text{sec} \iff$ The area is **decreasing** at a rate of 90 in^2/sec

$\implies \frac{dP}{dt} = -2 \text{ in/sec} \iff$ The perimeter is **decreasing** at a rate of 2 in/sec

EXAMPLE: A car traveling north at 60 mi/hr and a van traveling east at 40 mi/hr leave an intersection at the same time. At what rate will the distance between them be changing 4 hours later?

STEP 1: Interpret all given information, draw the relevant picture & label relevant variables:



$x \equiv$ "distance van traveled" $\quad y \equiv$ "distance car traveled" $\quad z \equiv$ "distance between car & van"

"car traveling north at 60 mi/hr" $\iff \dfrac{dy}{dt} = 60$ mi/hr $\quad$ "van traveling east at 40 mi/hr" $\iff \dfrac{dx}{dt} = 40$ mi/hr

"4 hours later" $\iff t = 4$ hr $\quad$ "At what rate will the distance between car & van be changing?" $\iff$ Find $\dfrac{dz}{dt}$

$x = t\dfrac{dx}{dt} = (4\text{ hr})(40\text{ mi/hr}) = 160\text{ mi} \qquad y = t\dfrac{dy}{dt} = (4\text{ hr})(60\text{ mi/hr}) = 240\text{ mi}$

STEP 2: Recognize the geometric or physical relationship among the variables:

Since the variables x, y, z are the sides of a right triangle, relate them via **Pythagorean's Theorem**: $z^2 = x^2 + y^2$

STEP $2^1/_2$: Solve for the unknown variable:

$x = 160$ mi and $y = 240$ mi $\implies z^2 = (160\text{ mi})^2 + (240\text{ mi})^2 \implies z = 80\sqrt{13}$ mi

STEP 3: Implicitly differentiate the relationship found in **STEP 2** with respect to time (t):

$$z^2 = x^2 + y^2 \implies \frac{d}{dt}\left[z^2\right] = \frac{d}{dt}\left[x^2 + y^2\right] \implies 2z\frac{dz}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$$

STEP 4: Plug in what you know and solve for what you don't know:

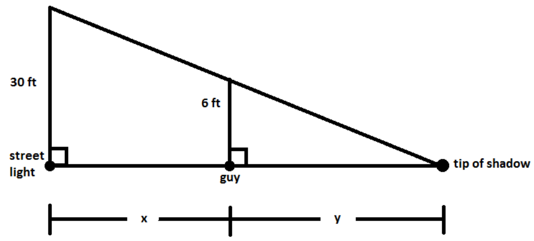
$$2(80\sqrt{13}\text{ mi})\frac{dz}{dt} = 2(160\text{ mi})(40\text{ mi/hr}) + 2(240\text{ mi})(60\text{ mi/hr}) \implies \frac{dz}{dt} = \frac{260}{\sqrt{13}}\text{ mi/hr} \approx 72.111\text{ mi/hr}$$

STEP 5: Interpret the answer:

$\dfrac{dz}{dt} \approx 72.111$ mi/hr $\iff$ The distance between the car & van is changing at about 72.111 mi/hr

EXAMPLE: A street light pole is 30 ft tall. A guy 6 ft tall walks away from the pole at 3 ft/sec along a straight path. How fast is the tip of his shadow moving away from the pole when he is 60 ft from the pole?

STEP 1: Interpret all given information, draw the relevant picture & label relevant variables:



$y \equiv$ length of guy's shadow $\implies \dfrac{dy}{dt} \equiv$ growth rate of guy's shadow

"guy walks away from the pole at 3 ft/sec" $\iff \dfrac{dx}{dt} = 3$ ft/sec "guy is 60 ft from the base of the pole" $\iff x = 60$ ft

"How fast is the tip of his shadow **moving away from the pole**?" $\iff$ Find $\dfrac{dx}{dt} + \dfrac{dy}{dt}$ $\left(\text{NOT } \dfrac{dy}{dt} \text{ – Do you see why??}\right)$

STEP 2: Recognize the geometric or physical relationship among the variables:

Notice we have two right triangles that have identical angles (do you see why the angles are identical?)

Recall from high school geometry that two triangles with identical angles are called **similar triangles**.

Therefore, we have the following relationship between x & y: $\dfrac{30 \text{ ft}}{x+y} = \dfrac{6 \text{ ft}}{y}$

STEP 3: Implicitly differentiate the relationship found in **STEP 2** with respect to time (t):

Rewrite the relationship to avoid the quotient rule: $\dfrac{30}{x+y} = \dfrac{6}{y} \iff 30y = 6(x+y) \iff 30y = 6x + 6y \iff 24y = 6x$

Now, implicitly differentiate w.r.t. time t: $24\dfrac{dy}{dt} = 6\dfrac{dx}{dt}$

STEP 4: Plug in what you know and solve for what you don't know:

$(24 \text{ ft})\dfrac{dy}{dt} = (6 \text{ ft})\dfrac{dx}{dt} \implies (24 \text{ ft})\dfrac{dy}{dt} = (6 \text{ ft})(3 \text{ ft/sec}) \implies \dfrac{dy}{dt} = \dfrac{3}{4}$ ft/sec

$\implies \dfrac{dx}{dt} + \dfrac{dy}{dt} = 3 \text{ ft/sec} + \dfrac{3}{4} \text{ ft/sec} = \dfrac{15}{4} \text{ ft/sec} = 3.75 \text{ ft/sec}$

STEP 5: Interpret the answer:

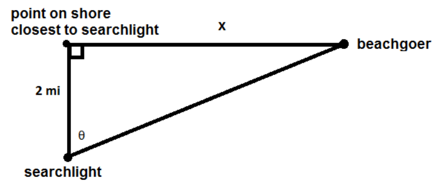
$\dfrac{dx}{dt} + \dfrac{dy}{dt} = 3.75$ ft/sec $\iff$ The tip of the guy's shadow is moving away from pole at 3.75 ft/sec

EXAMPLE: A revolving searchlight in a lighthouse 2 mi offshore is following a beachgoer along the shore.

When the beachgoer is 3 mi from the point closest to the searchlight, the searchlight is turning at the rate of 60°/hr.

How fast is the beachgoer walking at that moment?

STEP 1: Interpret all given information, draw the relevant picture & label relevant variables:



"beachgoer is 3 mi from the point closest to the searchlight" $\iff x = 3$ mi

$\theta \equiv$ angle from due north (**bearing**) of searchlight

"searchlight is turning at a rate of 60°/hr" $\iff \dfrac{d\theta}{dt} = 60°/\text{hr} = \dfrac{\pi}{3}$ radians/hr (Remember all angles must be in **radians**)

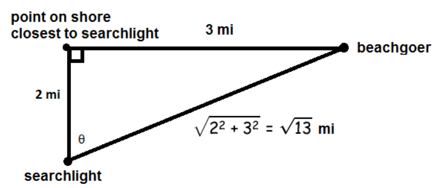
STEP 2: Recognize the geometric or physical relationship among the variables (AKA the **general situation**):

Since we have a right triangle, use trigonometry to relate the variables x and θ: $\tan\theta = \dfrac{x}{2\text{ mi}}$

STEP $2^1/_2$: Solve for the unknown variable:

$x = 3\text{ mi} \implies \tan\theta = \dfrac{3\text{ mi}}{2\text{ mi}} = \dfrac{3}{2}$ (Remember, trig functions are **ratios**, meaning they're **dimensionless**.)

Then the **specific situation** is:



Now establish the other 5 trigonometric functions of θ as one of them will be needed after differentiation.

(Doing this avoids introducing inverse trig functions which are harder to work with)

$\sin\theta = \dfrac{3}{\sqrt{13}}$ $\quad \csc\theta = \dfrac{\sqrt{13}}{3}$ $\quad \cos\theta = \dfrac{2}{\sqrt{13}}$ $\quad \sec\theta = \dfrac{\sqrt{13}}{2}$ $\quad \tan\theta = \dfrac{3}{2}$ $\quad \cot\theta = \dfrac{2}{3}$

STEP 3: Implicitly differentiate the relationship found in **STEP 2** with respect to time (t):

$\tan\theta = \dfrac{x}{2} \implies \dfrac{d}{dt}[\tan\theta] = \dfrac{d}{dt}\left[\dfrac{x}{2}\right] \implies \left(\sec^2\theta\right)\dfrac{d\theta}{dt} = \dfrac{1}{2}\dfrac{dx}{dt}$

STEP 4: Plug in what you know and solve for what you don't know:

$\left(\dfrac{\sqrt{13}}{2}\right)^2\left(\dfrac{\pi}{3}\text{ (radians)/hr}\right) = \dfrac{1}{2\text{ mi}}\dfrac{dx}{dt} \implies \dfrac{dx}{dt} = \dfrac{13\pi}{6}\text{ mi/hr} \approx 6.807\text{ mi/hr}$

STEP 5: Interpret the answer:

$\dfrac{dx}{dt} \approx 6.807\text{ mi/hr} \iff$ The beachgoer is walking at a rate of about 6.807 mi/hr