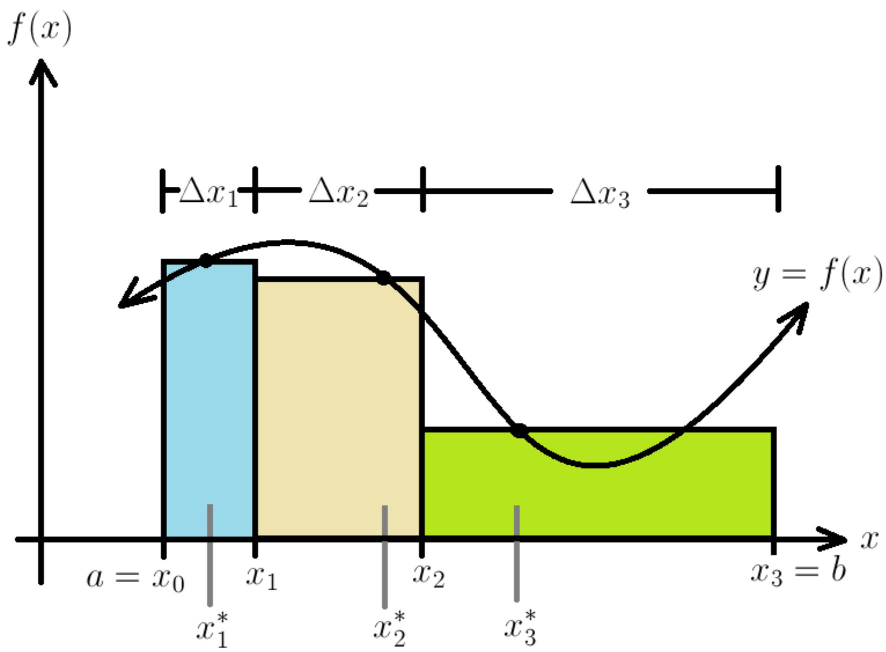
# RIEMANN SUMS

## NON-UNIFORM (GENERAL) RIEMANN SUMS:

- There are to be $N$ rectangles of **varying widths** spanning the **closed interval** $[a, b]$ on the $x$-axis.
- **Partition** of interval $[a, b]$ is denoted $\mathcal{P} := \{x_0, x_1, x_2, \ldots, x_{N-1}, x_N\}$, where $a = x_0 < x_1 < x_2 < \cdots < x_{N-1} < x_N = b$
- **Tags** of interval $[a, b]$ are denoted $\mathcal{T} := \{x_1^*, x_2^*, \ldots, x_{N-1}^*, x_N^*\}$, where each $x_k^* \in [x_{k-1}, x_k]$
- The **width** of the $k^{th}$ rectangle is denoted by $\Delta x_k := x_k - x_{k-1}$.
- The **norm** of the partition $\mathcal{P}$ is the **largest** rectangle width : $||\mathcal{P}|| := \max\{\Delta x_k : k = 1, 2, 3, \ldots, N-1, N\}$
- The **Riemann sum** of the function $f$ over interval $[a, b]$ using $N$ rectangles is defined by $\mathcal{S}_{(\mathcal{P},\mathcal{T})}(f) := \sum_{k=1}^{N} f(x_k^*)\Delta x_k$.




In the above illustrated example:

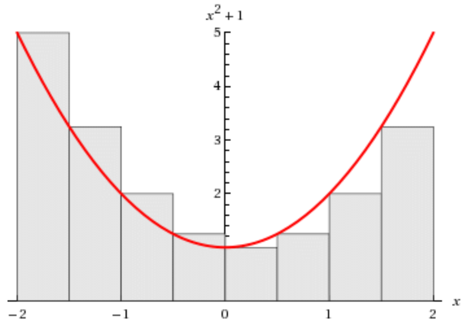
- # Rectangles : $N = 3$
- Partition : $\mathcal{P} = \{x_0, x_1, x_2, x_3\}$
- Tags : $\mathcal{T} = \{x_1^*, x_2^*, x_3^*\}$
- Norm of Partition : $||\mathcal{P}|| = \Delta x_3 := x_3 - x_2$
- Riemann Sum : $\mathcal{S}_{(\mathcal{P},\mathcal{T})}(f) = \sum_{k=1}^{N} f(x_k^*)\Delta x_k = f(x_1^*)\Delta x_1 + f(x_2^*)\Delta x_2 + f(x_3^*)\Delta x_3$

## UNIFORM (REGULAR) RIEMANN SUMS:

- There are to be $N$ rectangles of **equal width** spanning the **closed interval** $[a, b]$ on the $x$-axis.
- Hence, the **width** of each rectangle is the same and is denoted by $\Delta x := \dfrac{b-a}{N}$.
- Hence, the **partition** of interval $[a, b]$ is $\mathcal{P} := \{a, a + \Delta x, a + 2\Delta x, a + 3\Delta x, \ldots, a + (N-1)\Delta x, a + N\Delta x = b\}$
- **Tags** of interval $[a, b]$ are denoted $\mathcal{T} := \{x_1^*, x_2^*, \ldots, x_{N-1}^*, x_N^*\}$, where each $x_k^* \in [x_{k-1}, x_k]$ is **chosen the same way**.
- The **norm** of the partition $\mathcal{P}$ is $||\mathcal{P}|| := \Delta x = \dfrac{b-a}{N}$
- The **Riemann sum** of the function $f$ over interval $[a, b]$ using $N$ rectangles is defined by $\mathcal{S}_{\mathcal{T},N}(f) := \sum_{k=1}^{N} f(x_k^*)\Delta x$.

The five most common uniform Riemann sums are illustrated in the following two examples.

**EXAMPLE:** Let $f(x) = 1 + x^2$. Illustrate & compute the five basic uniform Riemann sums using 8 rectangles on $[-2, 2]$.




Left-Endpoint Riemann Sum:

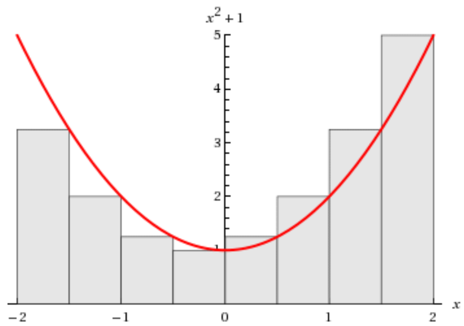
$\mathcal{P} = \{-2.0, -1.5, -1.0, -0.5, 0, 0.5, 1.0, 1.5, 2.0\}$

Rectangles : $N = 8$; Width : $\Delta x = 0.5$

Tags : $x_k^* = x_{k-1}$

$\mathcal{T} = \{-2.0, -1.5, -1.0, -0.5, 0, 0.5, 1.0, 1.5\}$

$$\mathcal{S}_{left,N}(f) = \sum_{k=1}^{N} f(x_k^*)\Delta x = 9.5$$




Right-Endpoint Riemann Sum:

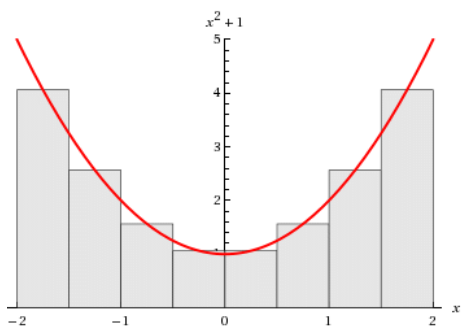
$\mathcal{P} = \{-2.0, -1.5, -1.0, -0.5, 0, 0.5, 1.0, 1.5, 2.0\}$

Rectangles : $N = 8$; Width : $\Delta x = 0.5$

Tags : $x_k^* = x_k$

$\mathcal{T} = \{-1.5, -1.0, -0.5, 0, 0.5, 1.0, 1.5, 2.0\}$

$$\mathcal{S}_{right,N}(f) = \sum_{k=1}^{N} f(x_k^*)\Delta x = 9.5$$




Mid-point Riemann Sum:

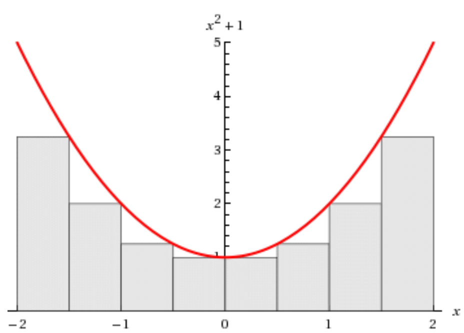
$\mathcal{P} = \{-2.0, -1.5, -1.0, -0.5, 0, 0.5, 1.0, 1.5, 2.0\}$

Rectangles : $N = 8$; Width : $\Delta x = 0.5$

Tags : $x_k^* = \frac{x_{k-1} + x_k}{2}$

$\mathcal{T} = \{-1.75, -1.25, -0.75, -0.25, 0.25, 0.75, 1.25, 1.75\}$

$$\mathcal{S}_{mid,N}(f) = \sum_{k=1}^{N} f(x_k^*)\Delta x = 9.25$$




Min-value Riemann Sum:

$\mathcal{P} = \{-2.0, -1.5, -1.0, -0.5, 0, 0.5, 1.0, 1.5, 2.0\}$

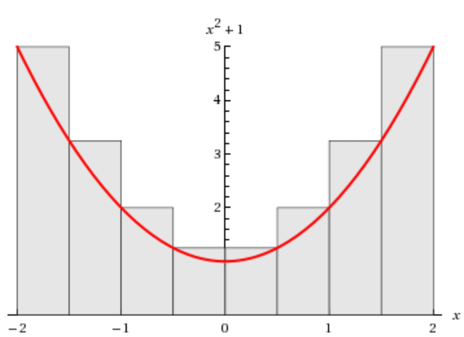
Rectangles : $N = 8$; Width : $\Delta x = 0.5$

$x_k^* = \arg\min\{f(x) : x \in [x_{k-1}, x_k]\}$

$\mathcal{T} = \{-1.5, -1.0, -0.5, 0, 0, 0.5, 1.0, 1.5\}$

**Lower Sum:**

$$\mathcal{LS}_N(f) = \sum_{k=1}^{N} f(x_k^*)\Delta x = 7.5$$




Max-value Riemann Sum:

$\mathcal{P} = \{-2.0, -1.5, -1.0, -0.5, 0, 0.5, 1.0, 1.5, 2.0\}$

Rectangles : $N = 8$; Width : $\Delta x = 0.5$

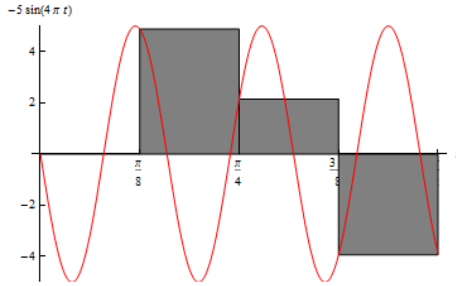
$x_k^* = \arg\max\{f(x) : x \in [x_{k-1}, x_k]\}$

$\mathcal{T} = \{-2.0, -1.5, -1.0, -0.5, 0.5, 1.0, 1.5, 2.0\}$

**Upper Sum:**

$$\mathcal{US}_N(f) = \sum_{k=1}^{N} f(x_k^*)\Delta x = 11.5$$

**EXAMPLE:** Let $g(t) = -5\sin(4\pi t)$. Illustrate & compute the five basic uniform Riemann sums using 4 rectangles on $\left[0, \frac{\pi}{2}\right]$.




Left-Endpoint Riemann Sum:

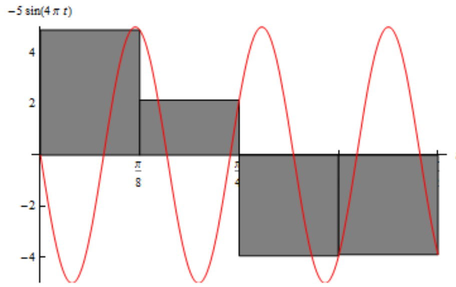
$\mathcal{P} = \left\{0, \frac{\pi}{8}, \frac{\pi}{4}, \frac{3\pi}{8}, \frac{\pi}{2}\right\}$

Rectangles : $N = 4$; Width : $\Delta t = \frac{\pi}{8}$

Tags : $t_k^* = t_{k-1}$

$\mathcal{T} = \left\{0, \frac{\pi}{8}, \frac{\pi}{4}, \frac{3\pi}{8}\right\}$

$$\mathcal{S}_{left,N}(g) = \sum_{k=1}^{N} g(t_k^*)\Delta t \approx 1.218$$




Right-Endpoint Riemann Sum:

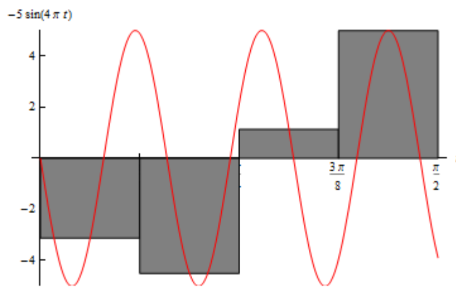
$\mathcal{P} = \left\{0, \frac{\pi}{8}, \frac{\pi}{4}, \frac{3\pi}{8}, \frac{\pi}{2}\right\}$

Rectangles : $N = 4$; Width : $\Delta t = \frac{\pi}{8}$

Tags : $t_k^* = t_k$

$\mathcal{T} = \left\{\frac{\pi}{8}, \frac{\pi}{4}, \frac{3\pi}{8}, \frac{\pi}{2}\right\}$

$$\mathcal{S}_{right,N}(g) = \sum_{k=1}^{N} g(t_k^*)\Delta t \approx -0.308$$




Mid-point Riemann Sum:

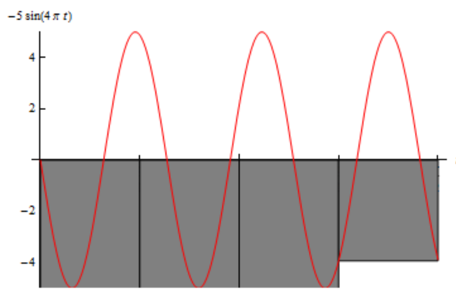
$\mathcal{P} = \left\{0, \frac{\pi}{8}, \frac{\pi}{4}, \frac{3\pi}{8}, \frac{\pi}{2}\right\}$

Rectangles : $N = 4$; Width : $\Delta t = \frac{\pi}{8}$

Tags : $t_k^* = \frac{t_{k-1}+t_k}{2}$

$\mathcal{T} = \left\{\frac{\pi}{16}, \frac{3\pi}{16}, \frac{5\pi}{16}, \frac{7\pi}{16}\right\}$

$$\mathcal{S}_{mid,N}(g) = \sum_{k=1}^{N} g(t_k^*)\Delta t \approx -0.582$$




Min-value Riemann Sum:

$\mathcal{P} = \left\{0, \frac{\pi}{8}, \frac{\pi}{4}, \frac{3\pi}{8}, \frac{\pi}{2}\right\}$

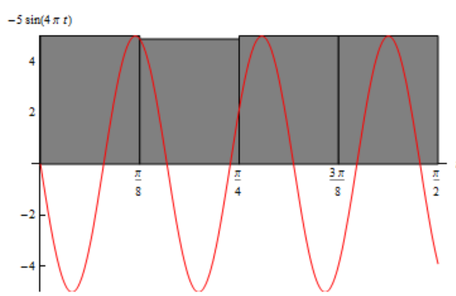
Rectangles : $N = 4$; Width : $\Delta t = \frac{\pi}{8}$

$t_k^* = \arg\min\{g(t) : t \in [t_{k-1}, t_k]\}$

$\mathcal{T} = \left\{\frac{1}{8}, \frac{5}{8}, \frac{9}{8}, \frac{3\pi}{8}\right\}$

**Lower Sum:**

$$\mathcal{LS}_N(g) = \sum_{k=1}^{N} g(t_k^*)\Delta t \approx -7.433$$




Max-value Riemann Sum:

$\mathcal{P} = \left\{0, \frac{\pi}{8}, \frac{\pi}{4}, \frac{3\pi}{8}, \frac{\pi}{2}\right\}$

Rectangles : $N = 4$; Width : $\Delta t = \frac{\pi}{8}$

$t_k^* = \arg\max\{g(t) : t \in [t_{k-1}, t_k]\}$

$\mathcal{T} = \left\{\frac{3}{8}, \frac{\pi}{8}, \frac{7}{8}, \frac{11}{8}\right\}$

**Upper Sum:**

$$\mathcal{US}_N(g) = \sum_{k=1}^{N} g(t_k^*)\Delta t \approx 7.806$$