Equations of Lines & Circles

DISTANCE BETWEEN TWO POINTS (x_0, y_0) **AND** (x_1, y_1) **:** $\sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2}$

EQUATION OF A (GENERAL) LINE:

- Assume line contains points (x_0, y_0), (x_1, y_1), **y-intercept** $(0, b)$, and **x-intercept** $(a, 0)$
- Then the **slope** is $m = \dfrac{y_1 - y_0}{x_1 - x_0}$
- Point-slope form : $y - y_0 = m(x - x_0)$
- Slope-intercept form : $y = mx + b$

EQUATION OF A HORIZONTAL LINE: $y = k$ (The slope of a horizontal line is zero)

EQUATION OF A VERTICAL LINE: $x = h$ (The slope of a vertical line DNE [**D**oes **N**ot **E**xist])

PARALLEL LINES:

- Two **vertical** lines are **parallel**.
- Two **non-vertical** lines ℓ_1, ℓ_2 are **parallel** $\iff \ell_1 \mid\mid \ell_2 \iff$ their slopes $m_1 = m_2$

PERPENDICULAR LINES: **slanted** line $\iff$ line that's neither horizontal nor vertical

- **Horizontal** lines are **perpendicular** to **vertical** lines.
- Two **slanted** lines ℓ_1, ℓ_2 are **perpendicular** $\iff \ell_1 \perp \ell_2 \iff$ their slopes $m_1 m_2 = -1$

INTERSECTION OF TWO LINES:

- Finding **intersection** of 2 lines $\ell_1, \ell_2 \iff$ solving a **system** of 2 linear equations
- 3 cases :
  (i) $\ell_1 \mid\mid \ell_2 \iff$ no solution (ii) ℓ_1, ℓ_2 coincident $\iff$ infinitely many solutions (iii) ℓ_1, ℓ_2 intersect $\iff$ one solution
- Solve system of 2 linear equations by using **substitution** : Solve one equation for either variable, then plug expression into same variable of other equation, then solve for other variable.

EXAMPLE: Suppose a line contains points $(1, 8)$ and $(4, -2)$. Determine its equation in slope-intercept form.

Let points $(x_0, y_0) = (1, 8)$ and $(x_1, y_1) = (4, -2)$. Then, slope $m = \dfrac{y_1 - y_0}{x_1 - x_0} = \dfrac{(-2) - (8)}{(4) - (1)} = -\dfrac{10}{3}$

Since we do not immediately recognize the y-intercept of this line, plug the values x_0, y_0, m into the point-slope form :

$$y - y_0 = m(x - x_0) \implies y - 8 = -\frac{10}{3}(x - 1) \implies y = 8 - \frac{10}{3}x + \frac{10}{3} \implies \boxed{y = -\frac{10}{3}x + \frac{34}{3}}$$

EXAMPLE: Suppose a line ℓ has x-intercept -10 and y-intercept 5. Determine its equation in slope-intercept form.

First, "line ℓ has x-intercept -10" $\iff$ "line ℓ contains point $(-10, 0)$" $\iff$ point $(-10, 0) \in \ell$

Next, "line ℓ has y-intercept 5" $\iff$ "line ℓ contains point $(0, 5)$" $\iff$ point $(0, 5) \in \ell$

Thus, let points $(x_0, y_0) = (-10, 0)$ and $(x_1, y_1) = (0, 5)$. Then, slope $m = \dfrac{y_1 - y_0}{x_1 - x_0} = \dfrac{(5) - (0)}{(0) - (-10)} = \dfrac{1}{2}$

Since we were given the y-intercept of line ℓ $(b = 5)$, plug the values m, b into the slope-intercept form :

$$\ell : y = mx + b \implies \boxed{\ell : y = \frac{1}{2}x + 5}$$

EXAMPLE: Let line ℓ_1 contain point $(-3, -12)$ and be perpendicular to $\ell_2 : -2x + y = 4$. Find slope-intercept form of ℓ_1.

First, $\ell_1 \perp \ell_2 \iff$ slopes $m_1 m_2 = -1$. Now, find slope m_2 of line ℓ_2: $-2x + y = 4 \implies y = 2x + 4 \implies m_2 = 2$

Hence, $m_1 m_2 = -1 \implies m_1(2) = -1 \implies m_1 = -\dfrac{1}{2} \implies$ slope of line ℓ_1 is $m_1 = -\dfrac{1}{2}$

Let $(x_0, y_0) = (-3, -12) \in \ell_1$. Then, plug the values x_0, y_0, m_1 into the point-slope form :

$$\ell_1 : y - y_0 = m_1(x - x_0) \implies \ell_1 : y - (-12) = -\frac{1}{2}(x - (-3)) \implies \ell_1 : y = -12 - \frac{1}{2}x - \frac{3}{2} \implies \boxed{\ell_1 : y = -\frac{1}{2}x - \frac{27}{2}}$$

EQUATION OF A CIRCLE:

- Assume circle has center (h,k) and radius $r>0$
- Standard form : $(x-h)^2+(y-k)^2=r^2$
- Complete the square of x^2+bx (where $b>0$) by adding term $\left(\frac{b}{2}\right)^2 \implies x^2+bx+\left(\frac{b}{2}\right)^2=\left(x+\frac{b}{2}\right)^2$
- Complete the square of x^2-bx (where $b>0$) by adding term $\left(\frac{b}{2}\right)^2 \implies x^2-bx+\left(\frac{b}{2}\right)^2=\left(x-\frac{b}{2}\right)^2$

EXAMPLE: Find the center & radius of the circle $x^2-2x+y^2+\sqrt{5}y-10=0$

The main task is to rewrite equation $x^2-2x+y^2+\sqrt{5}y-10=0$ in standard form $(x-h)^2+(y-k)^2=r^2$

First, move the constant -10 to the RHS of equation : $x^2-2x+y^2+\sqrt{5}y=10$

Complete the square in x : $x^2-2x+\left(\frac{-2}{2}\right)^2=x^2-2x+1=(x-1)^2$

Complete the square in y : $y^2+\sqrt{5}y+\left(\frac{\sqrt{5}}{2}\right)^2=y^2+\sqrt{5}y+\frac{5}{4}=\left(y+\frac{\sqrt{5}}{2}\right)^2$

Hence, we have $x^2-2x+[1-1]+y^2+\sqrt{5}y+\left[\frac{5}{4}-\frac{5}{4}\right]=10 \implies (x^2-2x+1)+\left(y^2+\sqrt{5}y+\frac{5}{4}\right)-1-\frac{5}{4}=10$

$\implies (x-1)^2+\left(y+\frac{\sqrt{5}}{2}\right)^2=10+1+\frac{5}{4} \implies (x-1)^2+\left(y+\frac{\sqrt{5}}{2}\right)^2=\frac{49}{4} \implies (x-1)^2+\left(y+\frac{\sqrt{5}}{2}\right)^2=\left(\frac{7}{2}\right)^2$

$\implies (x-1)^2+\left[y-\left(-\frac{\sqrt{5}}{2}\right)\right]^2=\left(\frac{7}{2}\right)^2$, which is in standard form.

Therefore, circle has its center at point $\left(1,-\frac{\sqrt{5}}{2}\right)$ and a radius of $\frac{7}{2}$

EXAMPLE: Find the center & radius of the circle $2x^2-12x+2y^2+8y-24=0$

The main task is to rewrite equation $2x^2-12x+2y^2+8y-24=0$ in standard form $(x-h)^2+(y-k)^2=r^2$

Notice if the LHS of standard form is expanded, we get $(x^2-2hx+h^2)+(y^2-2ky+k^2)=r^2$

Moreover, observe that the x^2 and y^2 terms of standard form have coefficient one,

so in our given equation, the x^2 and y^2 terms must also have coefficient one.

Hence, divide both sides of equation by 2 : $2x^2-12x+2y^2+8y-24=0 \implies x^2-6x+y^2+4y-12=0$

Now, move the constant -12 to the RHS of equation : $x^2-6x+y^2+4y=12$

Complete the square in x : $x^2-6x+\left(\frac{-6}{2}\right)^2=x^2-6x+9=(x-3)^2$

Complete the square in y : $y^2+4y+\left(\frac{4}{2}\right)^2=y^2+4y+4=(y+2)^2$

Hence, we have $x^2-6x+[9-9]+y^2+4y+[4-4]=12 \implies (x^2-6x+9)+(y^2+4y+4)-9-4=12$

$\implies (x-3)^2+(y+2)^2=12+9+4 \implies (x-3)^2+(y+2)^2=25$

$\implies (x-3)^2+[y-(-2)]^2=5^2$, which is in standard form.

Therefore, circle has its center at point $(3,-2)$ and a radius of 5

EXAMPLE: The equation $(x-1)^2+(y+10)^2=0$ does <u>not</u> describe a circle. Why not???

Well, this equation certainly looks like standard form $(x-h)^2+(y-k)^2=r^2$ with $h=1$ and $k=-10$,

but $r=0$, meaning the radius of this false circle is zero! We know circles must have **positive** radii (that is, $r>0$).

In fact, the graph of $(x-1)^2+(y+10)^2=0$ is a **single point** at $(h,k)=(1,-10)$!!