ESSENTIAL TRIGONOMETRY (I): TRIG FUNCTIONS

ANGLES:

- An **angle** consists of two **rays** with a common **vertex**.
- Angles are measured in : (i) degrees, (ii) degrees-minutes-seconds (D-M-S), or (iii) radians (preferred).
- Typical variables for angles : $\theta, \alpha, \beta, \varphi, A, B, C$
- A **positive angle** θ in **standard position** is swept **counter-clockwise** starting from the **positive** x**-axis**.
- Conversion of degrees $\to$ radians : (radians) $= \frac{\pi}{180^\circ} \times$(degrees)
- Conversion of radians $\to$ degrees : (degrees) $= \frac{180^\circ}{\pi} \times$(radians)
- Conversion of D-M-S $\to$ degrees : (degrees) = (degrees) $+ \frac{\text{(minutes)}}{60} + \frac{\text{(seconds)}}{3600}$
- Co-terminal angles: Conversion of general angle $\hat{\theta}$ to standard angle $\theta \in [0^\circ, 360^\circ)$: $\theta = \begin{cases} \hat{\theta} \bmod 360^\circ & \text{, if } \hat{\theta} \geq 0 \\ \left(\hat{\theta} \bmod 360^\circ\right) + 360^\circ & \text{, if } \hat{\theta} < 0 \end{cases}$
- NOTE: $R = P \bmod D \iff \frac{P}{D} = Q + \frac{R}{D}$ (i.e. **modular division** cares about the **remainder**, not the quotient.)

Point-wise (x-y-r) Definitions of Trig Functions: $(x^2 + y^2 = r^2$, where $r > 0)$

- Applies to all four Quadrants.
- $\sin\theta := \frac{y}{r}$, $\cos\theta := \frac{x}{r}$, $\tan\theta := \frac{y}{x}$
- $\csc\theta := \frac{r}{y}$, $\sec\theta := \frac{r}{x}$, $\cot\theta := \frac{x}{y}$

Right-triangle (acute) Definitions of Trig Functions:

- Applies only to Quadrant I & right triangles.
- $\sin\theta := \frac{\text{OPP}}{\text{HYP}}$, $\cos\theta := \frac{\text{ADJ}}{\text{HYP}}$, $\tan\theta := \frac{\text{OPP}}{\text{ADJ}}$
- $\csc\theta := \frac{\text{HYP}}{\text{OPP}}$, $\sec\theta := \frac{\text{HYP}}{\text{ADJ}}$, $\cot\theta := \frac{\text{ADJ}}{\text{OPP}}$
- Useful Mnemonic Device : 'sOH-cAH-tOA'

Special Right Triangles: (i) $45^\circ - 45^\circ - 90^\circ$ (ii) $30^\circ - 60^\circ - 90^\circ$

Reference Angles (degrees): QI: $\theta_r = \theta$ QII: $\theta_r = 180^\circ - \theta$ QIII: $\theta_r = \theta - 180^\circ$ QIV: $\theta = 360^\circ - \theta$

Reference Angles (radians): QI: $\theta_r = \theta$ QII: $\theta_r = \pi - \theta$ QIII: $\theta_r = \theta - \pi$ QIV: $\theta = 2\pi - \theta$

Positive Trig Functions: QI: All 6 fcns > 0 QII: $\sin\theta > 0, \csc\theta > 0$ QIII: $\tan\theta > 0, \cot\theta > 0$ QIV: $\cos\theta > 0, \sec\theta > 0$

Essential Trig Identities: Identities hold for every $\theta, A, B \in [0, 2\pi)$

- (Reciprocal) $\csc\theta = \dfrac{1}{\sin\theta}$, $\sec\theta = \dfrac{1}{\cos\theta}$, $\cot\theta = \dfrac{1}{\tan\theta}$ (Quotient) $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$, $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$
- (Pythagorean) $\sin^2\theta + \cos^2\theta = 1$, $1 + \tan^2\theta = \sec^2\theta$, $1 + \cot^2\theta = \csc^2\theta$
- (Negative-Angle) $\sin(-\theta) = -\sin\theta$, $\cos(-\theta) = \cos\theta$, $\tan(-\theta) = -\tan\theta$
- (Co-function) $\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$, $\csc\left(\frac{\pi}{2} - \theta\right) = \sec\theta$, $\tan\left(\frac{\pi}{2} - \theta\right) = \cot\theta$
- (Sum) $\sin(A+B) = \sin A\cos B + \cos A\sin B$, $\cos(A+B) = \cos A\cos B - \sin A\sin B$, $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A\tan B}$
- (Double-Angle) $\sin(2\theta) = 2\sin\theta\cos\theta$, $\cos(2\theta) = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$, $\tan(2\theta) = \dfrac{2\tan\theta}{1 - \tan^2\theta}$
- (Half-Angle) $\sin(\theta/2) = \pm\sqrt{\dfrac{1-\cos\theta}{2}}$, $\cos(\theta/2) = \pm\sqrt{\dfrac{1+\cos\theta}{2}}$, $\tan(\theta/2) = \pm\sqrt{\dfrac{1-\cos\theta}{1+\cos\theta}} = \dfrac{\sin\theta}{1+\cos\theta} = \dfrac{1-\cos\theta}{\sin\theta}$
  REMARK: The choice of the sign preceding the square root depends on which quadrant the angle $\theta/2$ resides.

SIMPLIFYING TRIG EXPRESSIONS: Use trig identities and/or write expression in terms of $\sin\theta$ and $\cos\theta$

EXAMPLE: (a) Convert $245°$ to radians (b) Convert $\frac{7\pi}{6}$ (radians) to degrees (c) Convert $12°20'40''$ (D-M-S) to degrees

(a) $245°\left(\frac{\pi}{180°}\right) = \boxed{\frac{49\pi}{36}}$ (b) $\frac{7\pi}{6}\left(\frac{180°}{\pi}\right) = \boxed{210°}$ (c) $12°20'40'' = 12° + \frac{20}{60} + \frac{40}{3600} \approx \boxed{12.3444°}$ (rounded)

EXAMPLE: (a) Convert $1700°$ to co-terminal standard position (b) Convert $-1350°$ to co-terminal standard position

(a) $1700° \bmod 360° = \boxed{260°}$ (b) $(-1350° \bmod 360°) + 360° = -270° + 360° = \boxed{90°}$

EXAMPLE: Find the reference angle for: (a) $75°$ (b) $\frac{3\pi}{4}$ (c) $220°$ (d) $\frac{13\pi}{8}$

(a) $75° \in (0°, 90°) =$ QI $\implies \theta_r = \boxed{75°}$ (b) $\frac{3\pi}{4} \in \left(\frac{\pi}{2}, \pi\right) =$ QII $\implies \theta_r = \pi - \frac{3\pi}{4} = \boxed{\frac{\pi}{4}}$

(c) $220° \in (180°, 270°) =$ QIII $\implies \theta_r = 220° - 180° = \boxed{40°}$ (d) $\frac{13\pi}{8} \in \left(\frac{3\pi}{2}, 2\pi\right) =$ QIV $\implies \theta_r = 2\pi - \frac{13\pi}{8} = \boxed{\frac{3\pi}{8}}$

EXAMPLE: Let θ be the angle whose terminal side passes thru the point $(-2, 2\sqrt{2})$. Find all six trig functions EXACTLY.

$x = -2, y = 2\sqrt{2} \implies r^2 = (-2)^2 + \left(2\sqrt{2}\right)^2 \implies r = \sqrt{12} = 2\sqrt{3}$

$$\implies \boxed{\sin\theta = \frac{y}{r} = \frac{\sqrt{6}}{3},\quad \cos\theta = \frac{x}{r} = -\frac{1}{\sqrt{3}},\quad \tan\theta = \frac{y}{x} = -\sqrt{2},\quad \csc\theta = \frac{3}{\sqrt{6}},\quad \sec\theta = -\sqrt{3},\quad \cot\theta = -\frac{1}{\sqrt{2}}}$$

EXAMPLE: Find all six trig functions of θ EXACTLY if: (a) $\theta = 45°$ (b) $\theta = \frac{4\pi}{3}$

(a) Since $\theta = 45°$, look at the $45°$-$45°$-$90°$ right triangle $\implies$ OPP $= 1$, ADJ $= 1$, HYP $= \sqrt{2}$

$$\implies \boxed{\sin\theta = \frac{\text{OPP}}{\text{HYP}} = \frac{1}{\sqrt{2}},\quad \cos\theta = \frac{\text{ADJ}}{\text{HYP}} = \frac{1}{\sqrt{2}},\quad \tan\theta = \frac{\text{OPP}}{\text{ADJ}} = 1,\quad \csc\theta = \sqrt{2},\quad \sec\theta = \sqrt{2},\quad \cot\theta = 1}$$

(b) $\frac{4\pi}{3} \in \left(\pi, \frac{3\pi}{2}\right) =$ QIII $\implies \theta_r = \frac{4\pi}{3} - \pi = \frac{\pi}{3} = 60°$

Since $\theta_r = 60°$, look at the $30°$-$60°$-$90°$ right triangle $\implies$ OPP $= \sqrt{3}$, ADJ $= 1$, HYP $= 2$

Now, $\theta \in$ QIII $\implies \sin\theta < 0, \cos\theta < 0, \tan\theta > 0$

$$\implies \boxed{\sin\theta = -\frac{\text{OPP}}{\text{HYP}} = -\frac{\sqrt{3}}{2},\quad \cos\theta = -\frac{\text{ADJ}}{\text{HYP}} = -\frac{1}{2},\quad \tan\theta = \frac{\text{OPP}}{\text{ADJ}} = \sqrt{3},\quad \csc\theta = -\frac{2}{\sqrt{3}},\quad \sec\theta = -2,\quad \cot\theta = \frac{1}{\sqrt{3}}}$$

EXAMPLE: Find EXACTLY: (a) $\sin 75°$ (b) $\tan\left(-\frac{\pi}{6}\right)$ (c) $\cos\left(\frac{\pi}{8}\right)$ (d) $\cos 8° \cos 82° - \sin 8° \sin 82°$

(a) $\sin 75° = \sin(30° + 45°) = \sin 30° \cos 45° + \cos 30° \sin 45° = \left(\frac{1}{2}\right)\left(\frac{1}{\sqrt{2}}\right) + \left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{\sqrt{2}}\right) = \boxed{\frac{\sqrt{2}+\sqrt{6}}{4}}$

(b) $\tan\left(-\frac{\pi}{6}\right) = -\tan\left(\frac{\pi}{6}\right) = -\tan 30° = \boxed{-\frac{1}{\sqrt{3}}}$

(c) $\cos\left(\frac{\pi}{8}\right) = \cos 22.5° = \cos\left(\frac{45°}{2}\right) = \pm\sqrt{\frac{1+\cos 45°}{2}} = \pm\sqrt{\frac{1+\frac{1}{\sqrt{2}}}{2}} = \pm\sqrt{\frac{1+\frac{\sqrt{2}}{2}}{2}} = \pm\sqrt{\frac{2+\sqrt{2}}{4}} = \boxed{\frac{\sqrt{2+\sqrt{2}}}{2}}$

(d) $\cos 8° \cos 82° - \sin 8° \sin 82° = \cos(8° + 82°) = \cos 90° = \boxed{0}$

EXAMPLE: Simplify: (a) $\csc\theta(1-\cos\theta)(1+\cos\theta)$ (b) $\frac{\sec\theta + \csc\theta}{1+\tan\theta}$ (c) $\frac{1+\sec\theta}{\sec\theta - 1} - \frac{1+\cos\theta}{1-\cos\theta}$

(a) $\csc\theta(1-\cos\theta)(1+\cos\theta) = \csc\theta(1-\cos^2\theta) = \csc\theta\sin^2\theta = \frac{\sin^2\theta}{\sin\theta} = \boxed{\sin\theta}$

(b) $\frac{\sec\theta + \csc\theta}{1+\tan\theta} = \frac{\frac{1}{\cos\theta} + \frac{1}{\sin\theta}}{1 + \frac{\sin\theta}{\cos\theta}} = \frac{\sin\theta + \cos\theta}{\sin\theta\cos\theta + \sin^2\theta} = \frac{\sin\theta + \cos\theta}{\sin\theta(\sin\theta + \cos\theta)} = \frac{1}{\sin\theta} = \boxed{\csc\theta}$

(c) $\frac{1+\sec\theta}{\sec\theta-1} - \frac{1+\cos\theta}{1-\cos\theta} = \frac{(1+\sec\theta)(1-\cos\theta) - (1+\cos\theta)(\sec\theta-1)}{(\sec\theta-1)(1-\cos\theta)} = \frac{1-\cos\theta+\sec\theta-1-\sec\theta+1-1+\cos\theta}{(\sec\theta-1)(1-\cos\theta)} = \boxed{0}$