ESSENTIAL TRIGONOMETRY (II): INVERSE TRIG FUNCTIONS

INVERSE TRIG FUNCTIONS: (To avoid confusion with the reciprocal, write 'arctrg x' instead of 'trg^{-1}x')

- $\theta = \arcsin x \iff \sin\theta = x$ where $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] = \text{QI} \cup \text{QIV}$
- $\theta = \arccos x \iff \cos\theta = x$ where $\theta \in [0,\pi] = \text{QI} \cup \text{QII}$
- $\theta = \arctan x \iff \tan\theta = x$ where $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \subset \text{QI} \cup \text{QIV}$
- $\theta = \text{arccsc } x \iff \csc\theta = x$ where $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \setminus \{0\} \subset \text{QI} \cup \text{QIV}$
- $\theta = \text{arcsec } x \iff \sec\theta = x$ where $\theta \in [0,\pi] \setminus \left\{\frac{\pi}{2}\right\} \subset \text{QI} \cup \text{QII}$
- $\theta = \text{arccot } x \iff \cot\theta = x$ where $\theta \in (0,\pi) \subset \text{QI} \cup \text{QII}$

EXAMPLE: Find EXACTLY: (a) $\arccos\frac{1}{2}$ (b) $\arctan(-1)$ (c) $\text{arccsc}(-\sqrt{2})$

(a) $\theta = \arccos\frac{1}{2} \iff \cos\theta = \frac{1}{2}$ where $\theta \in [0,\pi] \implies \theta = 60° = \boxed{\frac{\pi}{3}}$

(b) $\theta = \arctan(-1) \iff \tan\theta = -1$ where $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \implies \theta = -45° = \boxed{-\frac{\pi}{4}}$

(c) $\theta = \text{arccsc }(-\sqrt{2}) \iff \csc\theta = -\sqrt{2}$ where $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]\setminus\{0\} \implies \sin\theta = -\frac{1}{\sqrt{2}}$ where $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]\setminus\{0\} \implies \theta = \boxed{-\frac{\pi}{4}}$

EXAMPLE: Find EXACTLY: (a) $\arcsin(-3.5)$ (b) $\text{arcsec}\left(-\frac{1}{7}\right)$ (c) $\text{arccsc } 0$

(a) $\theta = \arcsin(-3.5) \iff \sin\theta = -3.5$ where $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, but $-3.5 \notin \text{Rng}(\sin\theta)$. Thus, $\theta \in \emptyset \implies \boxed{\arcsin(-3.5) \text{ is undefined}}$

(b) $\theta = \text{arcsec } \left(-\frac{1}{7}\right) \iff \sec\theta = -\frac{1}{7}$ where $\theta \in [0,\pi]\setminus\left\{\frac{\pi}{2}\right\} \implies \cos\theta = -7 \notin \text{Rng}(\cos\theta) \implies \boxed{\text{arcsec}\left(-\frac{1}{7}\right) \text{ is undefined}}$

(c) $\theta = \text{arccsc } 0 \iff \csc\theta = 0$ where $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \setminus \{0\} \implies \frac{1}{\sin\theta} = 0 \implies 1 = 0 \implies \boxed{\text{arccsc } 0 \text{ is undefined}}$

EXAMPLE: Find EXACTLY: (a) $\tan\left(\arcsin\left(-\frac{1}{\sqrt{2}}\right)\right)$ (b) $\cos\left(\arcsin\frac{1}{4} + \arccos\left(-\frac{2}{5}\right)\right)$ (c) $\csc\left(2 \text{ arccot }\left(-\frac{1}{2}\right)\right)$

(a) $\theta = \arcsin\left(-\frac{1}{\sqrt{2}}\right) \implies \sin\theta = -\frac{1}{\sqrt{2}}$ where $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \implies \theta \in \text{QIV} \implies \left\{\begin{array}{l} y = -1 \\ r = \sqrt{2} \end{array}\right\} \implies x = 1$

Hence, $\tan\left(\arcsin\left(-\frac{1}{\sqrt{2}}\right)\right) = \tan\theta := \frac{y}{x} = \frac{-1}{1} = \boxed{-1}$

(b) $A = \arcsin\frac{1}{4} \implies \sin A = \frac{1}{4}$ where $A \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \implies A \in \text{QI} \implies \left\{\begin{array}{l} y_A = 1 \\ r_A = 4 \end{array}\right\} \implies x_A = \sqrt{15}$

$B = \arccos\left(-\frac{2}{5}\right) \iff \cos B = -\frac{2}{5}$ where $B \in [0,\pi] \implies B \in \text{QII} \implies \left\{\begin{array}{l} x_B = -2 \\ r_B = 5 \end{array}\right\} \implies y_B = \sqrt{21}$

$\implies \cos(A+B) = \cos A\cos B - \sin A \sin B := \left(\frac{x_A}{r_A}\right)\left(\frac{x_B}{r_B}\right) - \left(\frac{y_A}{r_A}\right)\left(\frac{y_B}{r_B}\right) = \left(\frac{\sqrt{15}}{4}\right)\left(\frac{-2}{5}\right) - \left(\frac{1}{4}\right)\left(\frac{\sqrt{21}}{5}\right) = \boxed{-\frac{2\sqrt{15}+\sqrt{21}}{20}}$

(c) $\theta = \text{arccot } \left(-\frac{1}{2}\right) \implies \cot\theta = -\frac{1}{2}$ where $\theta \in (0,\pi) \implies \theta \in \text{QII} \implies \left\{\begin{array}{l} x = -1 \\ y = 2 \end{array}\right\} \implies r = \sqrt{5}$

Hence, $\csc\left(2 \text{ arccot }\left(-\frac{1}{2}\right)\right) = \csc(2\theta) = \frac{1}{\sin(2\theta)} = \frac{1}{2\sin\theta\cos\theta} := \frac{1}{2\left(\frac{y}{r}\right)\left(\frac{x}{r}\right)} = \frac{1}{2\left(\frac{2}{\sqrt{5}}\right)\left(\frac{-1}{\sqrt{5}}\right)} = \boxed{-\frac{5}{4}}$