CONTINUITY OF FUNCTIONS

NOTATION FOR CONTINUITY:

- A function f is **continuous at a point** $x = p \iff f \in C^0(\{p\})$
- A function f is **continuous on a set** $S \iff f \in C^0(S) \iff \forall p \in S,\ f \in C^0(\{p\})$
- A function f is **continuous on a closed interval** $[a, b] \iff f \in C^0[a, b] \iff \forall p \in [a, b],\ f \in C^0(\{p\})$
- A function f is **continuous on an open interval** $(a, b) \iff f \in C^0(a, b) \iff \forall p \in (a, b),\ f \in C^0(\{p\})$
- A function f is **continuous everywhere** $\iff f \in C^0(-\infty, \infty) \iff f \in C^0(\mathbb{R})$

DEFINITION OF CONTINUITY: $(p \in \mathbb{R})$

- $f \in C^0(\{p\}) \iff \Big[f(p) \text{ exists} \quad \text{AND} \quad \lim_{x \to p} f(x) \text{ exists} \quad \text{AND} \quad \lim_{x \to p} f(x) = f(p)\Big]$
- In plain English: If x is 'near' p, then $f(x)$ must be 'near' $f(p)$
- A function that is **not continuous** at point $x = p$ is said to have a **discontinuity** at $x = p$

CONTINUITY RULES: $(k, p \in \mathbb{R})$

- (C.0) (Constants) $f(x) = k \implies f \in C^0(\mathbb{R})$
- (C.1) (Polynomials) f is a **polynomial** $\implies f \in C^0(\mathbb{R})$
- (C.2) (Elementary Fcns) f is an **elementary function** $\implies f \in C^0\big(\mathrm{Dom}(f)\big)$
- (C.3) (Multiple Rule) $f \in C^0(\{p\}) \implies kf \in C^0(\{p\})$
- (C.4) (Sum/Diff Rule) $f, g \in C^0(\{p\}) \implies f \pm g \in C^0(\{p\})$
- (C.5) (Product Rule) $f, g \in C^0(\{p\}) \implies fg \in C^0(\{p\})$
- (C.6) (Quotient Rule) $f, g \in C^0(\{p\})$ AND $g(p) \neq 0 \implies f/g \in C^0(\{p\})$
- (C.7) (Composition Rule) $g \in C^0(\{p\})$ AND $f \in C^0(\{g(p)\}) \implies f \circ g \in C^0(\{p\})$

COMPOSITION LIMIT RULE: $(p \in \mathbb{R})$

- $\Big[\lim_{x \to p} g(x) = L \quad \text{AND} \quad f \in C^0(\{L\})\Big] \implies \lim_{x \to p} f[g(x)] = f\Big(\lim_{x \to p} g(x)\Big) = f(L)$
- What this means: If the outer function f of composition $f \circ g$ is continuous at $x = p$, then the **limit** as x approaches p can be **passed inside** the outer function f.

ONE-SIDED CONTINUITY: $(a, b \in \mathbb{R} \text{ s.t. } a < b)$

- A function f is **right-continuous** at $a \iff f \in C^+(\{a\}) \iff \lim_{x \to a^+} f(x) = f(a)$
- A function f is **left-continuous** at $b \iff f \in C^-(\{b\}) \iff \lim_{x \to b^-} f(x) = f(b)$
- Relationship to '2-sided continuity': $f \in C^-(\{p\})$ AND $f \in C^-(\{p\}) \iff f \in C^0(\{p\})$
- Checking 1-sided continuity is only necessary for determining if a **piecewise function** is continuous on a **closed interval**: $f \in C^0(a, b)$ AND $f \in C^+(\{a\})$ AND $f \in C^-(\{b\}) \iff f \in C^0[a, b]$
- 1-sided continuity shows up in junior-level **probability** (MATH 3342) and senior-level **analysis** (MATH 4350) courses.

TYPES OF DISCONTINUITY: See Strauss pg 70 for visual examples of these discontinuities.

- **Removable**: Either $\Big[f(c) \text{ DNE AND } \lim_{x \to c} f(x) \in \mathbb{R}\Big]$ OR $\Big[f(c) \text{ exists AND } \lim_{x \to c} f(x) \in \mathbb{R} \text{ AND } \lim_{x \to c} f(x) \neq f(c)\Big]$
- **Jump**: Both 1-sided limits are finite & unequal. i.e., $\lim_{x \to c^-} f(x) \in \mathbb{R}$ AND $\lim_{x \to c^+} f(x) \in \mathbb{R}$ AND $\lim_{x \to c^-} f(x) \neq \lim_{x \to c^+} f(x)$
- **Break**: At least one 1-sided limit is infinite. i.e., $\Big[\lim_{x \to c^-} f(x) = -\infty \text{ or } +\infty\Big]$ AND/OR $\Big[\lim_{x \to c^+} f(x) = -\infty \text{ or } +\infty\Big]$

EXAMPLE: Determine the interval(s) where $f(x) = x^6 - 2x^3 - 3x^2 - 4x - 5$ is continuous.

Observe that f is a **polynomial** (and, thus, **elementary**) $\implies$ $\text{Dom}(f) = \mathbb{R} \overset{C.2}{\implies}$ $\boxed{f \in C^0(\mathbb{R}) \iff f \in C^0(-\infty, \infty)}$

EXAMPLE: Determine the interval(s) where $g(t) = \dfrac{1}{t^5 - t^4 - 12t^3}$ is continuous.

Observe that g is a **rational function** and, thus, **elementary**.

Since g is a rational function, first factor the denominator: $t^5 - t^4 - 12t^3 = t^3(t^2 - t - 12) = t^3(t-4)(t+3)$

Next, set denominator equal to zero & solve for t: $t^5 - t^4 - 12t^3 = 0 \implies t^3(t-4)(t+3) = 0 \implies t \in \{-3, 0, 4\}$

Hence, $\text{Dom}(g) = \mathbb{R} \setminus \{t \in \mathbb{R} : t^5 - t^4 - 12t^3 = 0\} = \mathbb{R} \setminus \{-3, 0, 4\} = (-\infty, -3) \cup (-3, 0) \cup (0, 4) \cup (4, \infty)$

$\overset{C.2}{\implies} g \in C^0\Big(\text{Dom}(g)\Big) \implies \boxed{g \in C^0\Big(\mathbb{R} \setminus \{-3, 0, 4\}\Big) \iff g \in C^0\Big((-\infty, -3) \cup (-3, 0) \cup (0, 4) \cup (4, \infty)\Big)}$

REMARK: We say g has **discontinuities** at the t-values $-3, 0$, and 4.

EXAMPLE: (a) Where is $h(z) = \dfrac{8 - 2z^2}{z^2 + 5z + 6}$ continuous? (b) Identify the type of each discontinuity.

Observe that h is a **rational function** and, thus, **elementary**, so factor numerator & denominator:

$$h(z) = \frac{8 - 2z^2}{z^2 + 5z + 6} = \frac{2(4 - z^2)}{(z+2)(z+3)} = \frac{2(2-z)(2+z)}{(z+2)(z+3)}$$

Hence, $\text{Dom}(h) = \mathbb{R} \setminus \{t \in \mathbb{R} : z^2 + 5z + 6 = 0\} = \mathbb{R} \setminus \{-3, -2\} = (-\infty, -3) \cup (-3, -2) \cup (-2, \infty)$

$\overset{C.2}{\implies} h \in C^0\Big(\text{Dom}(h)\Big) \implies \boxed{h \in C^0\Big(\mathbb{R} \setminus \{-3, -2\}\Big) \iff h \in C^0\Big((-\infty, -3) \cup (-3, -2) \cup (-2, \infty)\Big)}$

(b) From part (a), the two discontinuities of h occur at $z = -3$ and $z = -2$

To determine the **type of discontinuity** at $z = -3$, compute $h(-3)$, $\lim\limits_{z \to -3} h(z)$, $\lim\limits_{z \to (-3)^-} h(z)$, and $\lim\limits_{z \to (-3)^+} h(z)$:

$$h(-3) = DNE, \ \lim_{z \to -3} h(z) = \lim_{z \to -3} \frac{2(2-z)(2+z)}{(z+2)(z+3)} \overset{L.3}{=} \left[\lim_{z \to -3} 2(2-z)(2+z)\right] \left[\lim_{z \to -3} \frac{1}{z+2}\right] \left[\lim_{z \to -3} \frac{1}{z+3}\right]$$

$$\overset{NS}{=} [2(2 - (-3))(2 + (-3))] \left[\frac{1}{(-3)+2}\right] \left[\lim_{z \to -3} \frac{1}{z+3}\right] = 10 \lim_{z \to -3} \frac{1}{z+3} \overset{CV}{=} 10 \lim_{u \to 0} \frac{1}{u} = DNE$$

$$\lim_{z \to (-3)^-} h(z) = \lim_{z \to (-3)^-} \frac{2(2-z)(2+z)}{(z+2)(z+3)} \overset{NS}{=} 10 \lim_{z \to (-3)^-} \frac{1}{z+3} \overset{CV}{=} 10 \lim_{u \to 0^-} \frac{1}{u} \overset{S.1}{=} 10(-\infty) \overset{E.4}{=} -\infty$$

Therefore, since $\lim\limits_{z \to (-3)^-} h(z) = -\infty$, $\boxed{h \text{ has a } \textbf{break} \text{ discontinuity at } z = -3}$

To determine the **type of discontinuity** at $z = -2$, compute $h(-2)$, $\lim\limits_{z \to -2} h(z)$, $\lim\limits_{z \to (-2)^-} h(z)$, and $\lim\limits_{z \to (-2)^+} h(z)$:

$$h(-2) = DNE, \ \lim_{z \to -2} h(z) \overset{NS}{=} \frac{2(2 - (-2))(2 + (-2))}{((-2)+2)((-2)+3)} = \frac{0}{0} \implies \text{Rewrite/simplify function (by factoring)}$$

$$\implies \lim_{z \to -2} h(z) = \lim_{z \to -2} \frac{2(2-z)(2+z)}{(z+2)(z+3)} = \lim_{z \to -2} \frac{2(2-z)}{z+3} \overset{NS}{=} \frac{2(2 - (-2))}{(-2)+3} = 8$$

Therefore, since $h(-2) = DNE$ and $\lim\limits_{z \to -2} h(z) \in \mathbb{R}$, $\boxed{h \text{ has a } \textbf{removable (hole)} \text{ discontinuity at } z = -2}$

EXAMPLE: Let $v(t) = \begin{cases} t+1 & , \text{if } t < 5 \\ t^3 & , \text{if } t \geq 5 \end{cases}$. (a) Is $v \in C^0(\{5\})$? (Justify) (b) If not, what type of discontinuity occurs?

(a) $\lim\limits_{t \to 5^-} v(t) = (5) + 1 = 6$ and $\lim\limits_{t \to 5^+} v(t) = (5)^3 = 125$

Since $\lim\limits_{t \to 5^-} v(t) \neq \lim\limits_{t \to 5^+} v(t)$, $\boxed{v \text{ is NOT continuous at } t = 5}$

(b) The fact that both 1-sided limits are finite but unequal means, by definition,

that a $\boxed{\textbf{jump} \text{ discontinuity}}$ occurs at $t = 5$.

EXAMPLE: Let $T(x) = \begin{cases} \cos(3x) & , \text{if } x < \pi \\ -1 & , \text{if } x = \pi \\ \sin\left(\frac{3}{2}x\right) & , \text{if } x > \pi \end{cases}$. Is T continuous at $x = \pi$? (Justify)

$\lim\limits_{x \to \pi^-} T(x) = \cos(3\pi) = \cos(\pi) = -1$, $\lim\limits_{x \to \pi^+} T(x) = \sin\left(\dfrac{3}{2}\pi\right) = -1$, $T(\pi) = -1$

$\implies \lim\limits_{x \to \pi} T(x) = -1$. Thus, since $T(\pi)$ exists, $\lim\limits_{x \to \pi} T(x)$ exists, and $\lim\limits_{x \to \pi} T(x) = T(\pi)$, $\boxed{T \text{ is continuous at } x = \pi}$