THE FUNDAMENTAL THEOREMS OF CALCULUS

DEFINITION OF THE INTEGRAL:

- A function f is **integrable** on interval $[a, b]$ if: (i) $[a, b] \subseteq$ Dom(f) AND (ii) $\displaystyle\int_a^b f(x)\ dx := \lim_{||\mathcal{P}|| \to 0} \sum_{k=1}^{N} f(x_k^*)\Delta x_k$ exists.
- The **definite integral** of $f(x)$ is denoted $\displaystyle\int_a^b f(x)\ dx$ and is read: "The integral from a to b of $f(x)$ with respect to x"
- THEOREM: f is **continuous** on interval $[a, b] \implies f$ is **integrable** on interval $[a, b]$.
- Geometrically, if $f(x) \geq 0\ \forall x \in [a, b]$, then $\displaystyle\int_a^b f(x)\ dx$ is the **<u>total area</u> under the curve** f bounded by the x-axis.

  Otherwise, if $f(x) \leq 0$ for some $x \in [a, b]$, then $\displaystyle\int_a^b f(x)\ dx$ is the **<u>net area</u> under the curve** f bounded by the x-axis.

ONE-DIMENSIONAL MOTION OF A PARTICLE:

- Given **velocity function** $v(t)$, the **total distance** traveled from time $t = t_0$ to $t = t_1$ is $\displaystyle\int_{t_0}^{t_1} |v(t)|\ dt$.
- Given **velocity function** $v(t)$, the **net displacement** traveled from time $t = t_0$ to $t = t_1$ is $\displaystyle\int_{t_0}^{t_1} v(t)\ dt$.

FIRST FUNDAMENTAL THEOREM OF CALCULUS (1^{st} F-T-C):

- (FTC.1) Let $f \in C[a, b]$ and $F'(x) = f(x)\ \forall x \in [a, b]$. Then, $\displaystyle\int_a^b f(x)\ dx = \Big[F(x)\Big]_{x=a}^{x=b} = F(b) - F(a)$.
- Let f be a **continuous** function on interval $[a, b]$ and F be an **antiderivative** of f. Then, $\displaystyle\int_a^b f(x)\ dx = F(b) - F(a)$.
- This theorem is "fundamental" to calculus as it makes a connection between differentiation & integration.
- The beauty of this theorem is that it allows computation of definite integrals **without using Riemann sums**.

SECOND FUNDAMENTAL THEOREM OF CALCULUS (2^{nd} F-T-C):

- (FTC.2) Let $f \in C[a, b]$. Then, $\displaystyle\frac{d}{dx}\int_a^x f(t)\ dt = f(x) \quad \forall x \in [a, b]$
- (Generalisation) Let $f, g' \in C[a, b]$. Then, $\displaystyle\frac{d}{dx}\int_a^{g(x)} f(t)\ dt = f\left[g(x)\right] g'(x)$
- The beauty of this theorem is that it works even when the integral is **nonelementary**.

BASIC DEFINITE INTEGRAL RULES:

Here, $-\infty < a < c < b < \infty$ and f, g are integrable.

- (DINT.0) (Degenerate Interval) $\displaystyle\int_a^a f(x)\ dx = 0$
- (DINT.1) (Flip Interval Rule) $\displaystyle\int_b^a f(x)\ dx = -\int_a^b f(x)\ dx$
- (DINT.2) (Lump Interval Rule) $\displaystyle\int_a^c f(x)\ dx + \int_c^b f(x)\ dx = \int_a^b f(x)\ dx$
- (DINT.3) (Multiple Rule) $\displaystyle\int_a^b kf(x)\ dx = k\int_a^b f(x)\ dx \qquad \Big(\text{where } k \in \mathbb{R}\Big)$
- (DINT.4) (Sum/Diff Rule) $\displaystyle\int_a^b \left[f(x) \pm g(x)\right]\ dx = \int_a^b f(x)\ dx \pm \int_a^b g(x)\ dx$
- (DINT.5) (Dominance Rule) Suppose $f(x) \leq g(x)$ for $x \in [a, b]$. Then, $\displaystyle\int_a^b f(x)\ dx \leq \int_a^b g(x)\ dx$.
- (DINT.6) (Even Function) f is **even** $\implies \displaystyle\int_{-a}^a f(x)\ dx = 2\int_0^a f(x)\ dx \qquad \Big[f$ is **even** $\iff f(-x) = f(x)\Big]$
- (DINT.7) (Odd Function) f is **odd** $\implies \displaystyle\int_{-a}^a f(x)\ dx = 0 \qquad \Big[f$ is **odd** $\iff f(-x) = -f(x)\Big]$

EXAMPLE: Evaluate $\int_0^{\pi/4} \sin\theta \, d\theta$.

$$\int_0^{\pi/4} \sin\theta \, d\theta \overset{INT.8}{=} \Big[-\cos\theta + C\Big]_{\theta=0}^{\theta=\pi/4} \overset{FTC.1}{=} [-\cos(\pi/4)+C] - [-\cos(0)+C] = -\frac{1}{\sqrt{2}} + 1 + (C - C) = \boxed{1 - \frac{1}{\sqrt{2}} \approx 0.2929}$$

Notice that the **constant of integration** C cancels out when using the F-T-C.

So going forward, C will never to added to anti-derivatives in definite integrals.

EXAMPLE: Evaluate $\int_0^1 \left(x^4 - 2x^3 + 1\right) \, dx$.

$$\int_0^1 \left(x^4 - 2x^3 + 1\right) dx \overset{DINT.4}{=} \int_0^1 x^4 \, dx - \int_0^1 2x^3 \, dx + \int_0^1 dx \overset{INT.4}{=} \left[\frac{1}{5}x^5\right]_{x=0}^{x=1} - \left[\frac{1}{2}x^4\right]_{x=0}^{x=1} + \Big[x\Big]_{x=0}^{x=1}$$

$$\overset{FTC.1}{=} \left[\frac{1}{5}(1)^5 - \frac{1}{5}(0)^5\right] - \left[\frac{1}{2}(1)^4 - \frac{1}{2}(0)^4\right] + [(1) - (0)] = \boxed{\frac{7}{10}}$$

EXAMPLE: Find the area under the curve $f(x) = |x|$ bounded by the x-axis and the vertical lines $x = -5$ & $x = 8$.

Recall that $|x|$ is a **piecewise function**: $|x| := \begin{cases} x & \text{, if } x \geq 0 \\ -x & \text{, if } x < 0 \end{cases}$

$$\Longrightarrow \quad \text{Area} = \int_{-5}^{8} f(x)\,dx = \int_{-5}^{8} |x|\,dx \overset{DINT.2}{=} \int_{-5}^{0} -x\,dx + \int_0^8 x\,dx \overset{INT.4}{=} \left[-\frac{1}{2}x^2\right]_{x=-5}^{x=0} + \left[\frac{1}{2}x^2\right]_{x=0}^{x=8}$$

$$\overset{FTC.1}{=} \left[-\frac{1}{2}(0)^2 - \left(-\frac{1}{2}(-5)^2\right)\right] + \left[\frac{1}{2}(8)^2 - \frac{1}{2}(0)^2\right] = \boxed{\frac{89}{2}}$$

EXAMPLE: Find $\frac{d}{dx}\int_x^4 t^3 \, dt$.

Since it's possible to find the anti-derivative of t^3, there are two approaches one can take:

(Method 1: Using the 1^{st} F-T-C) $\frac{d}{dx}\int_x^4 t^3\,dt \overset{INT.4}{=} \frac{d}{dx}\left[\frac{1}{4}t^4\right]_{t=x}^{t=4} \overset{FTC.1}{=} \frac{d}{dx}\left[\frac{1}{4}(4)^4 - \frac{1}{4}(x)^4\right] = \frac{d}{dx}\left[64 - \frac{1}{4}x^4\right] = \boxed{-x^3}$

(Method 2: Using the 2^{nd} F-T-C) $\frac{d}{dx}\int_x^4 t^3\,dt \overset{DINT.1}{=} -\frac{d}{dx}\int_4^x t^3\,dt \overset{FTC.2}{=} \boxed{-x^3}$

EXAMPLE: Find $\frac{d}{dx}\int_1^{x^2} 8t^7 \, dt$.

Since it's possible to find the anti-derivative of $8t^7$, there are two approaches one can take:

(Method 1: Using the 1^{st} F-T-C) $\frac{d}{dx}\int_1^{x^2} 8t^7\,dt \overset{INT.4}{=} \frac{d}{dx}\left[t^8\right]_{t=1}^{t=x^2} \overset{FTC.1}{=} \frac{d}{dx}\left[\left(x^2\right)^8 - (1)^8\right] = \frac{d}{dx}\left[x^{16} - 1\right] = \boxed{16x^{15}}$

(Method 2: Using the 2^{nd} F-T-C) $\frac{d}{dx}\int_1^{x^2} 8t^7\,dt \overset{FTC.2}{=} 8(x^2)^7 \frac{d}{dx}\left[x^2\right] = \left(8x^{14}\right)(2x) = \boxed{16x^{15}}$

EXAMPLE: Find $\frac{d}{dw}\int_2^w \sin\left(u^2\right) \, du$.

Since the integral is **nonelementary**, it's not possible to find its exact antiderivative, hence use the 2^{nd} F-T-C:

$$\frac{d}{dw}\int_2^w \sin\left(u^2\right) du \overset{FTC.2}{=} \boxed{\sin\left(w^2\right)}$$

EXAMPLE: Use the Dominance Rule to determine an **upper bound** of $\int_0^{\pi/2} 2x\sin x \, dx$.

The integral requires a more sophisticated integration technique to be learned in Calculus II.

Recall that $\text{Rng}(\sin x) = [-1, 1] \Longrightarrow \sin x \leq 1 \Longrightarrow 2x\sin x \leq 2x$

$$\overset{DINT.5}{\Longrightarrow} \int_0^{\pi/2} 2x\sin x\,dx \leq \int_0^{\pi/2} 2x\,dx \overset{INT.4}{=} \left[x^2\right]_{x=0}^{x=\pi/2} \overset{FTC.1}{=} \left(\frac{\pi}{2}\right)^2 - (0)^2 = \frac{\pi^2}{4}$$

Thus, $\boxed{\int_0^{\pi/2} 2x\sin x\,dx \leq \frac{\pi^2}{4} \approx 2.4674}$.

NOTE: The exact value of the integral is $\int_0^{\pi/2} 2x\sin x\,dx = 2$, so the upper bound is a reasonably tight estimate.