AREA BETWEEN TWO CURVES [SST 6.1]

- AREAS OF **VERTICALLY SIMPLE (V-SIMPLE) REGIONS**
  - SETUP: Given $f, g \in C[a,b]$ s.t. $f(x) \geq g(x) \quad \forall x \in [a,b]$ $\Big($i.e. curve f lies above curve g over $[a,b]\Big)$
    Let R be the region bounded by curves $y = f(x)$, $y = g(x)$, and the lines $x = a$, $x = b$.
  - TASK: Find the area of region R.
  - Let **partition** $\mathcal{P} := \{x_0, x_1, x_2, \ldots, x_{N-1}, x_N\} \subset [a,b]$ be **arbitrary**.
  - Let **tags** $\mathcal{T} := \{x_1^*, x_2^*, x_3^*, \ldots, x_{N-1}^*, x_N^*\} \subset [a,b]$ be **arbitrary**.
  - **Key element**: **V-Rectangle** (V-Rect)

Width	of	k^{th} V-Rect	:=	$\Big($Length of k^{th} subinterval$\Big)$	=	Δx_k
Height	of	k^{th} V-Rect	:=	(Top BC) − (Bottom BC)	=	$f(x_k^*) - g(x_k^*)$
Area	of	k^{th} V-Rect	:=	$\Big($Height$\Big) \times \Big($Width$\Big)$	=	$\Big[f(x_k^*) - g(x_k^*)\Big]\Delta x_k$

  - Riemann Sum: Area(R) $\approx A_N^* := \displaystyle\sum_{k=1}^{N} \Big[f(x_k^*) - g(x_k^*)\Big]\Delta x_k$
  - Integral: Area(R) $= \displaystyle\lim_{N\to\infty} A_N^* = \int_{\text{smallest x-coord.}}^{\text{largest x-coord.}} \Big[f(x) - g(x)\Big]\,dx = \boxed{\int_a^b \Big[f(x) - g(x)\Big]\,dx}$

- AREAS OF **HORIZONTALLY SIMPLE (H-SIMPLE) REGIONS**
  - SETUP: Given $p, q \in C[c,d]$ s.t. $p(y) \geq q(y) \quad \forall y \in [c,d]$ $\Big($i.e. curve p lies to the right of curve q over $[c,d]\Big)$
    Let R be the region bounded by curves $x = p(y)$, $x = q(y)$, and the lines $y = c$, $y = d$.
  - TASK: Find the area of region R.
  - Let **partition** $\mathcal{P} := \{y_0, y_1, y_2, \ldots, y_{N-1}, y_N\} \subset [c,d]$ be **arbitrary**.
  - Let **tags** $\mathcal{T} := \{y_1^*, y_2^*, y_3^*, \ldots, y_{N-1}^*, y_N^*\} \subset [c,d]$ be **arbitrary**.
  - **Key element**: **H-Rectangle** (H-Rect)

Width	of	k^{th} H-Rect	:=	$\Big($Length of k^{th} subinterval$\Big)$	=	Δy_k
Length	of	k^{th} H-Rect	:=	(Right BC) − (Left BC)	=	$p(y_k^*) - q(y_k^*)$
Area	of	k^{th} H-Rect	:=	$\Big($Length$\Big) \times \Big($Width$\Big)$	=	$\Big[p(y_k^*) - q(y_k^*)\Big]\Delta y_k$

  - Riemann Sum: Area(R) $\approx A_N^* := \displaystyle\sum_{k=1}^{N} \Big[p(y_k^*) - q(y_k^*)\Big]\Delta y_k$
  - Integral: Area(R) $= \displaystyle\lim_{N\to\infty} A_N^* = \int_{\text{smallest y-coord.}}^{\text{largest y-coord.}} \Big[p(y) - q(y)\Big]\,dy = \boxed{\int_c^d \Big[p(y) - q(y)\Big]\,dy}$

- **PROCEDURE FOR CHOOSING APPROPRIATE KEY ELEMENT:**
  - STEP 1: Sketch region R
  - STEP 2: Characterize region R:
    * Label all **boundary curves (BC's)** of region R (both in terms of x and in terms of y)
    * Label all **boundary points (BP's)**, which are the **intersection points** of the BC's.
  - STEP 3: Determine the simplicity of region R:

REGION	**KEY ELEMENT**
V-Simple	V-Rect
H-Simple	H-Rect
Both	Choose either V-Rect or H-Rect
Neither	Subdivide Region along a BP & repeat STEP 3 for each subregion

EX 6.1.1: Let R be the region bounded by the curve $y = \ln x$, the x-axis, and lines $x = 1$, $x = 10$.

(a) Sketch & characterize region R.

(b) Setup Riemann sum to estimate Area(R) **using V-Rects**.

(c) Setup integral to find Area(R) **using V-Rects**.

(d) Setup Riemann sum to estimate Area(R) **using H-Rects**.

(e) Setup integral to find Area(R) **using H-Rects**.

EX 6.1.2: Let R be the region bounded by curves $y = x^2$, $y = x$, the y-axis, and the line $x = 2$.

(a) Sketch & characterize region R.

(b) Setup integral to find Area(R) **using V-Rects**.

(c) Setup integral to find Area(R) **using H-Rects**.

EX 6.1.3: Let R be the region bounded by curves $y = x^2$ and $y = \sqrt[3]{x}$.

(a) Sketch & characterize region R.

(b) Setup integral to find Area(R) **using V-Rects**.

(c) Setup integral to find Area(R) **using H-Rects**.

EX 6.1.4: Let R be the region bounded by the curve $x = 1 + \sqrt{y}$, the y-axis, and lines $y = 1$, $y = 4$.

(a) Sketch & characterize region R.

(b) Setup integral to find Area(R) **using V-Rects**.

(c) Setup integral to find Area(R) **using H-Rects**.

EX 6.1.5: Let R be the region bounded by curves $x = 2 - y^2$ and $x = -y$.

(a) Sketch & characterize region R.

(b) Setup integral to find Area(R) **using V-Rects**.

(c) Setup integral to find Area(R) **using H-Rects**.

EX 6.1.6: Let R be the region bounded by the curve $y = \sin(2x)$, the x-axis and lines $x = \pi/2$, $x = 3\pi/2$.

(a) Sketch & characterize region R.

(b) Setup integral to find Area(R) **using V-Rects**.

EX 6.1.7: Let R be the region bounded by the curve $y = \arctan x$ and lines $x = -1$, $x = \frac{1}{\sqrt{3}}$, $y = \pi$.

(a) Sketch & characterize region R.

(b) Setup integral to find Area(R) **using V-Rects**.