VOLUMES OF SOLIDS: CROSS SECTIONS, WASHERS, SHELLS [SST 6.2]

- VOLUMES OF SOLIDS WITH **V-SIMPLE BASE** & CROSS SECTIONS PERPENDICULAR ($\perp$) TO **X-AXIS**:
  - SETUP: Given $f, g \in C[a,b]$ s.t. $f(x) \geq g(x) \quad \forall x \in [a,b]$ $\Big($i.e. curve f lies above curve g over $[a,b]\Big)$
    Let R be the region bounded by curves $y = f(x)$, $y = g(x)$, and the lines $x = a$, $x = b$.
    Let S be the solid with base R and cross sections $\perp$ to the x-**axis** with area $A(x)$.
  - TASK: Find the volume of the solid S with base R.
  - Let **partition** $\mathcal{P} := \{x_0, x_1, x_2, \ldots, x_{N-1}, x_N\} \subset [a,b]$ be **arbitrary**.
    Let **tags** $\mathcal{T} := \{x_1^*, x_2^*, x_3^*, \ldots, x_{N-1}^*, x_N^*\} \subset [a,b]$ be **arbitrary**.
  - **Key element**: **V-Slab** (i.e. slab standing up)

Thickness	of	k^{th} V-Slab	:=	$\Big($Length of k^{th} subinterval$\Big)$	=	Δx_k
Base Area	of	k^{th} V-Slab	:=	(Area of cross section)	=	$A(x_k^*)$
Volume	of	k^{th} V-Slab	:=	$\Big($Base Area$\Big) \times \Big($Thickness$\Big)$	=	$A(x_k^*)\,\Delta x_k$

  - Riemann Sum: Volume(S) $\approx V_N^* := \displaystyle\sum_{k=1}^{N} A(x_k^*)\,\Delta x_k$
  - Integral: Volume(S) $= \displaystyle\lim_{N\to\infty} V_N^* = \int_{\text{smallest x-coord.}}^{\text{largest x-coord.}} A(x)\,dx = \boxed{\int_a^b A(x)\,dx}$

- VOLUMES OF SOLIDS WITH **H-SIMPLE BASE** & CROSS SECTIONS PERPENDICULAR ($\perp$) TO **Y-AXIS**:
  - SETUP: Given $p, q \in C[c,d]$ s.t. $p(y) \geq q(y) \quad \forall y \in [c,d]$ $\Big($i.e. curve p lies to the right of curve q over $[c,d]\Big)$
    Let R be the region bounded by curves $x = p(y)$, $x = q(y)$, and the lines $y = c$, $y = d$.
    Let S be the solid with base R and cross sections $\perp$ to the y-**axis** with area $A(y)$.
  - TASK: Find the volume of the solid S with base R.
  - Let **partition** $\mathcal{P} := \{y_0, y_1, y_2, \ldots, y_{N-1}, y_N\} \subset [c,d]$ be **arbitrary**.
    Let **tags** $\mathcal{T} := \{y_1^*, y_2^*, y_3^*, \ldots, y_{N-1}^*, y_N^*\} \subset [c,d]$ be **arbitrary**.
  - **Key element**: **H-Slab** (i.e. slab lying down)

Thickness	of	k^{th} H-Slab	:=	$\Big($Length of k^{th} subinterval$\Big)$	=	Δy_k
Base Area	of	k^{th} H-Slab	:=	(Area of cross section)	=	$A(y_k^*)$
Volume	of	k^{th} H-Slab	:=	$\Big($Base Area$\Big) \times \Big($Thickness$\Big)$	=	$A(y_k^*)\,\Delta y_k$

  - Riemann Sum: Volume(S) $\approx V_N^* := \displaystyle\sum_{k=1}^{N} A(y_k^*)\,\Delta y_k$
  - Integral: Volume(S) $= \displaystyle\lim_{N\to\infty} V_N^* = \int_{\text{smallest y-coord.}}^{\text{largest y-coord.}} A(y)\,dy = \boxed{\int_c^d A(y)\,dy}$

- **PROCEDURE FOR CHOOSING APPROPRIATE KEY ELEMENT:**
  - STEP 1: Sketch region R
  - STEP 2: Characterize region R:
    * Label all **boundary curves (BC's)** of region R (both in terms of x and in terms of y)
    * Label all **boundary points (BP's)**, which are the **intersection points** of the BC's.
  - STEP 3: Determine the simplicity of region R:

REGION	**KEY ELEMENT**
V-Simple	V-Rect
H-Simple	H-Rect
Both	Choose either V-Rect or H-Rect
Neither	Subdivide Region along a BP & repeat STEP 3 for each subregion

- VOLUMES OF SOLIDS BY REVOLVING A **V-ALIGNED REGION** ABOUT **X-AXIS** USING **WASHERS**:

  - SETUP: Given $f, g \in C[a,b]$ s.t. $f(x) \geq g(x) \geq 0 \quad \forall x \in [a,b]$

    Let R be the region bounded by curves $y = f(x)$, $y = g(x)$, and the lines $x = a$, $x = b$.

    Let S be the solid formed by revolving region R about the x-axis.

  - TASK: Find the volume of the solid of revolution S about the x-axis using washers.

  - Let **partition** $\mathcal{P} := \{x_0, x_1, x_2, \ldots, x_{N-1}, x_N\} \subset [a,b]$

    Let **tags** $\mathcal{T} := \{x_1^*, x_2^*, x_3^*, \ldots, x_{N-1}^*, x_N^*\} \subset [a,b]$

  - **Key element**: **V-Washer** (i.e. washer standing up)

Thickness	of	k^{th} V-Washer	:=	$\big($Length of k^{th} subinterval$\big)$	=	Δx_k
Outer Radius	of	k^{th} V-Washer	:=	(Distance between farther BC & axis of rev.)	=	$f\left(x_k^*\right)$
Inner Radius	of	k^{th} V-Washer	:=	(Distance between closer BC & axis of rev.)	=	$g\left(x_k^*\right)$
Volume	of	k^{th} V-Washer	:=	$\pi \times \left[\left(\text{Outer R.}\right)^2 - \left(\text{Inner R.}\right)^2\right] \times \left(\text{Thick.}\right)$	=	$\pi\left(\left[f\left(x_k^*\right)\right]^2 - \left[g\left(x_k^*\right)\right]^2\right)\Delta x_k$

  - Riemann Sum: $\text{Volume}(S) \approx V_N^* := \sum_{k=1}^{N} \pi\left(\left[f\left(x_k^*\right)\right]^2 - \left[g\left(x_k^*\right)\right]^2\right)\Delta x_k$

  - Integral: $\text{Volume}(S) = \lim_{N\to\infty} V_N^* = \boxed{\int_a^b \pi\left(\left[f(x)\right]^2 - \left[g(x)\right]^2\right)\,dx}$

- VOLUMES OF SOLIDS BY REVOLVING A **H-ALIGNED REGION** ABOUT **Y-AXIS** USING **WASHERS**:

  - SETUP: Given $p, q \in C[c,d]$ s.t. $p(y) \geq q(y) \geq 0 \quad \forall y \in [c,d]$

    Let R be the region bounded by curves $x = p(y)$, $x = q(y)$, and the lines $y = c$, $y = d$.

    Let S be the solid formed by revolving region R about the y-axis.

  - TASK: Find the volume of the solid of revolution S about the y-axis using washers.

  - Let **partition** $\mathcal{P} := \{y_0, y_1, y_2, \ldots, y_{N-1}, y_N\} \subset [c,d]$

    Let **tags** $\mathcal{T} := \{y_1^*, y_2^*, y_3^*, \ldots, y_{N-1}^*, y_N^*\} \subset [c,d]$

  - **Key element**: **H-Washer** (i.e. washer lying down)

Thickness	of	k^{th} H-Washer	:=	$\big($Length of k^{th} subinterval$\big)$	=	Δy_k
Outer Radius	of	k^{th} H-Washer	:=	(Distance between farther BC & axis of rev.)	=	$p\left(y_k^*\right)$
Inner Radius	of	k^{th} H-Washer	:=	(Distance between closer BC & axis of rev.)	=	$q\left(y_k^*\right)$
Volume	of	k^{th} H-Washer	:=	$\pi \times \left[\left(\text{Outer R.}\right)^2 - \left(\text{Inner R.}\right)^2\right] \times \left(\text{Thick.}\right)$	=	$\pi\left(\left[p\left(y_k^*\right)\right]^2 - \left[q\left(y_k^*\right)\right]^2\right)\Delta y_k$

  - Riemann Sum: $\text{Volume}(S) \approx V_N^* := \sum_{k=1}^{N} \pi\left(\left[p\left(y_k^*\right)\right]^2 - \left[q\left(y_k^*\right)\right]^2\right)\Delta y_k$

  - Integral: $\text{Volume}(S) = \lim_{N\to\infty} V_N^* = \boxed{\int_c^d \pi\left(\left[p(y)\right]^2 - \left[q(y)\right]^2\right)\,dy}$

- **REMARKS:**

  - The same procedure for region R outlined in the 1^{st} page above applies here as well.

  - A **disk** is simply a washer without a hole $\Longrightarrow$ either the curve $g(x) = 0$ (x-axis) or the curve $q(y) = 0$ (y-axis).

  - If the axis of revolution is neither the x-axis nor the y-axis but rather $x = \alpha$ or $y = \beta$ where $\alpha, \beta \in \mathbb{R}$, then the **radii** of the k^{th} washer would be the distance from the respective curve to the axis of revolution.

- VOLUMES OF SOLIDS BY REVOLVING A **V-ALIGNED REGION** ABOUT **Y-AXIS** USING **SHELLS**:

  - SETUP: Given $f, g \in C[a,b]$ s.t. $f(x) \geq g(x) \quad \forall x \in [a,b]$

    Let R be the region bounded by curves $y = f(x)$, $y = g(x)$, and the lines $x = a$, $x = b$.

    Let S be the solid formed by revolving region R about the y-axis.
  - <u>TASK:</u> Find the volume of the solid of revolution S about the y-axis using shells.
  - Let **partition** $\mathcal{P} := \{x_0, x_1, x_2, \ldots, x_{N-1}, x_N\} \subset [a,b]$

    Let **tags** $\mathcal{T} := \{x_1^*, x_2^*, x_3^*, \ldots, x_{N-1}^*, x_N^*\} \subset [a,b]$
  - **Key element**: **V-Shell** (i.e. cylindrical shell standing up)

Thickness	of	k^{th} V-Shell	:=	(Length of k^{th} subinterval)	=	Δx_k
Radius	of	k^{th} V-Shell	:=	(Distance from V-shell to axis of revolution)	=	x_k^*
Height	of	k^{th} V-Shell	:=	(Top BC) − (Bottom BC)	=	$f(x_k^*) - g(x_k^*)$
Volume	of	k^{th} V-Shell	:=	$2\pi \times \Big(\text{Radius}\Big) \times \Big(\text{Height}\Big) \times \Big(\text{Thickness}\Big)$	=	$2\pi x_k^* \Big[f(x_k^*) - g(x_k^*)\Big] \Delta x_k$

  - Riemann Sum: $\text{Volume}(S) \approx V_N^* := \sum_{k=1}^{N} 2\pi x_k^* \Big[f(x_k^*) - g(x_k^*)\Big] \Delta x_k$
  - Integral: $\text{Volume}(S) = \lim_{N\to\infty} V_N^* = \boxed{\int_a^b 2\pi x \Big[f(x) - g(x)\Big]\, dx}$

- VOLUMES OF SOLIDS BY REVOLVING A **H-ALIGNED REGION** ABOUT **X-AXIS** USING **SHELLS**:

  - SETUP: Given $p, q \in C[c,d]$ s.t. $p(y) \geq q(y) \quad \forall y \in [c,d]$

    Let R be the region bounded by curves $x = p(y)$, $x = q(y)$, and the lines $y = c$, $y = d$.

    Let S be the solid formed by revolving region R about the x-axis.
  - <u>TASK:</u> Find the volume of the solid of revolution S about the x-axis using shells.
  -
  - Let **partition** $\mathcal{P} := \{y_0, y_1, y_2, \ldots, y_{N-1}, y_N\} \subset [c,d]$

    Let **tags** $\mathcal{T} := \{y_1^*, y_2^*, y_3^*, \ldots, y_{N-1}^*, y_N^*\} \subset [c,d]$
  - **Key element**: **H-Shell** (i.e. cylindrical shell standing up)

Thickness	of	k^{th} H-Shell	:=	(Length of k^{th} subinterval)	=	Δy_k
Radius	of	k^{th} H-Shell	:=	(Distance from H-shell to axis of revolution)	=	y_k^*
Length	of	k^{th} H-Shell	:=	(Right BC) − (Left BC)	=	$p(y_k^*) - q(y_k^*)$
Volume	of	k^{th} H-Shell	:=	$2\pi \times \Big(\text{Radius}\Big) \times \Big(\text{Height}\Big) \times \Big(\text{Thickness}\Big)$	=	$2\pi y_k^* \Big[p(y_k^*) - q(y_k^*)\Big] \Delta y_k$

  - Riemann Sum: $\text{Volume}(S) \approx V_N^* := \sum_{k=1}^{N} 2\pi y_k^* \Big[p(y_k^*) - q(y_k^*)\Big] \Delta y_k$
  - Integral: $\text{Volume}(S) = \lim_{N\to\infty} V_N^* = \boxed{\int_c^d 2\pi y \Big[p(y) - q(y)\Big]\, dy}$

- <u>**REMARKS:**</u>

  - The same procedure for region R outlined in the 1^{st} page above applies here as well.
  - If the axis of revolution is neither the x-axis nor the y-axis but rather $x = \alpha$ or $y = \beta$ where $\alpha, \beta \in \mathbb{R}$, then the **radius** of the k^{th} shell will be the distance from the shell to the axis of revolution.

EX 6.2.1: Let R be the region bounded by curves $y = 2 - x^2$, $y = x$, and the y-axis (with $x \geq 0$).

Let S be the solid with base R and **square** cross sections $\perp$ to the x-**axis** that protrude towards you.

(a) Sketch & characterize region R. (b) Setup integral(s) to find the volume of S.

EX 6.2.2: Let R be the region bounded by curves $y = 2 - x^2$, $y = x$, and the y-axis (with $x \geq 0$).

Let S be the solid with base R and **square** cross sections $\perp$ to the y-**axis** that protrude towards you.

(a) Sketch & characterize region R. (b) Setup integral(s) to find the volume of S.

EX 6.2.3: Let R be the region bounded by curves $y = 2 - x^2$, $y = x$, and the y-axis (with $x \geq 0$).

Let S be the solid with base R and **semicircular** cross sections $\perp$ to the x-**axis** that protrude towards you.

(a) Sketch & characterize region R. (b) Setup integral(s) to find the volume of S.

EX 6.2.4: Let R be the region bounded by curves $y = 2 - x^2$, $y = x$, and the y-axis (with $x \geq 0$).

Let S be the solid with base R and **semicircular** cross sections $\perp$ to the y-**axis** that protrude towards you.

(a) Sketch & characterize region R. (b) Setup integral(s) to find the volume of S.

EX 6.2.5: Let R be the region bounded by curves $y = 2 - x^2$, $y = x$, and the y-axis (with $x \geq 0$).

Let S be the solid formed by revolving region R about the y**-axis using washers**.

(a) Sketch & characterize region R & axis of revolution.

(b) Setup integral(s) to find the volume of S.

EX 6.2.6: Let R be the region bounded by curves $y = 2 - x^2$, $y = x$, and the y-axis (with $x \geq 0$).

Let S be the solid formed by revolving region R about the y**-axis using shells**.

(a) Sketch & characterize region R & axis of revolution.

(b) Setup integral(s) to find the volume of S.

EX 6.2.7: Let R be the region bounded by curves $y = 2 - x^2$, $y = x$, and the y-axis (with $x \geq 0$).

Let S be the solid formed by revolving region R about the x-**axis using washers**.

(a) Sketch & characterize region R & axis of revolution.

(b) Setup integral(s) to find the volume of S.

EX 6.2.8: Let R be the region bounded by curves $y = 2 - x^2$, $y = x$, and the y-axis (with $x \geq 0$).

Let S be the solid formed by revolving region R about the x-**axis using shells**.

(a) Sketch & characterize region R & axis of revolution.

(b) Setup integral(s) to find the volume of S.

EX 6.2.9: Let R be the region bounded by curves $y = 2 - x^2$, $y = x$, and the y-axis (with $x \geq 0$).

Let S be the solid formed by revolving region R about the **line** $x = -1$ **using washers**.

(a) Sketch & characterize region R & axis of revolution.

(b) Setup integral(s) to find the volume of S.

EX 6.2.10: Let R be the region bounded by curves $y = 2 - x^2$, $y = x$, and the y-axis (with $x \geq 0$).

Let S be the solid formed by revolving region R about the **line** $x = -1$ **using shells**.

(a) Sketch & characterize region R & axis of revolution.

(b) Setup integral(s) to find the volume of S.

EX 6.2.11: Let R be the region bounded by curves $y = 2 - x^2$, $y = x$, and the y-axis (with $x \geq 0$).

Let S be the solid formed by revolving region R about the **line** $y = 3$ **using washers**.

(a) Sketch & characterize region R & axis of revolution. (b) Setup integral(s) to find the volume of S.

EX 6.2.12: Let R be the region bounded by curves $y = 2 - x^2$, $y = x$, and the y-axis (with $x \geq 0$).

Let S be the solid formed by revolving region R about the **line** $y = 3$ **using shells**.

(a) Sketch & characterize region R & axis of revolution. (b) Setup integral(s) to find the volume of S.