# POLAR FORMS: [SST 6.3]

- **RECTANGULAR (CARTESIAN) COORDINATES:**
  - Form: $(x, y)$ where $x, y \in \mathbb{R}$
  - Origin: $(x, y) = (0, 0)$ (Notice the origin has a unique rectangular coordinate)
  - Coordinate $(x, y)$ is **unique**.
- **POLAR COORDINATES:**
  - Form: $(r, \theta)$ where $\theta \in \mathbb{R}$ is ALWAYS in radians and $r \in \mathbb{R}$ (Notice: $r$ can be negative)
  - Pole: $(r, \theta) = (0, \theta)$ (Notice there's no unique polar coordinate for the pole)
  - Coordinate $(r, \theta)$ is **NOT unique**.
    * $(-r, \theta) = (r, \theta + \pi)$
    * $(r, \theta) = (r, \theta + 2n\pi) = (-r, \theta + (2n + 1)\pi)$ where $n \in \mathbb{Z}$
    * e.g. $\left(2, \frac{7\pi}{4}\right) = \left(2, -\frac{\pi}{4}\right) = \left(-2, \frac{3\pi}{4}\right) = \left(-2, -\frac{5\pi}{4}\right)$
  - Polar $\to$ Rectangular: $x = r\cos\theta$, $y = r\sin\theta$
  - Rectangular $\to$ Polar: $r^2 = x^2 + y^2$, $\tan\theta = \dfrac{y}{x}$, provided $x \neq 0$
- **GRAPHING POLAR CURVE:** $r = f(\theta)$
  - First, graph $r = f(\theta)$ on the usual $xy$-plane where $x = \theta$ & $y = r$. $\Big($**rectangular plot**$\Big)$

    Use special angles for $\theta$ : $\left\{0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{3\pi}{2}, \frac{7\pi}{4}, 2\pi\right\}$

    If $f(\theta)$ has a trig fcn, set its argument to these special angles and solve for $\theta$:
    * e.g. $f(\theta) = 7\sin(2\theta) \implies 2\theta \in \left\{0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{3\pi}{2}, \frac{7\pi}{4}, 2\pi\right\} \implies \theta \in \left\{0, \frac{\pi}{8}, \frac{\pi}{4}, \frac{3\pi}{8}, \frac{\pi}{2}, \frac{5\pi}{8}, \frac{3\pi}{4}, \frac{7\pi}{8}, \pi\right\}$
  - Next, use the rectangular plot of $r = f(\theta)$ to **trace** the polar graph of $r = f(\theta)$.
  - IMPORTANT: Except for equations of lines, **connect the dots using smooth curves**, not line segments!
- **POLAR CURVES:** $(a, b, c \in \mathbb{R} \setminus \{0\}, n \in \mathbb{Z}_+)$
  - EITHER GRAPH OR CONVERT TO RECTANGULAR FORM:
    * Lines thru Pole (Origin): $\theta = c$
    * Circles: $r = c$, $r = a\cos\theta$, $r = a\sin\theta$
  - ALWAYS CONVERT TO RECTANGULAR FORM (ALWAYS!):
    * Horizontal Lines off-pole: $r = a\csc\theta$
    * Vertical Lines off-pole: $r = a\sec\theta$
  - ALWAYS GRAPH (ALWAYS!):
    * Cardioids: $r = a \pm a\cos\theta$, $r = a \pm a\sin\theta$
    * Limaçons: $r = b \pm a\cos\theta$, $r = b \pm a\sin\theta$ ($b = a \implies$ cardioid)
    * Roses: $r = a\cos(n\theta)$, $r = a\sin(n\theta)$ ($n = 1 \implies$ circle)
    * Lemniscates: $r^2 = a^2\cos(2\theta)$, $r^2 = a^2\sin(2\theta)$
- **INTERSECTION OF TWO POLAR CURVES $r = f(\theta)$ & $r = g(\theta)$:**
  - Solving $f(\theta) = g(\theta)$ for $\theta$ finds some, but not necessarily all, intersection points.
  - In particular, intersections at the **pole** (origin) are nearly impossible to find algebraically because the pole has no single representation in polar coordinates that satisfies both $r = f(\theta)$ & $r = g(\theta)$.
  - Therefore, to find all intersection points, GRAPH BOTH CURVES!

- **AREA OF RADIALLY SIMPLE POLAR REGION W/ POLE AS A BP: (FROM FIRST PRINCIPLES):**
  - SETUP: Given function $f \in C[\alpha, \beta]$ s.t. $f(\theta)$ **never changes sign** $\forall \theta \in [\alpha, \beta]$ and $0 < \beta - \alpha \leq 2\pi$.
    Let $D$ be the region bounded by polar curve $r = f(\theta)$ and rays $\theta = \alpha$, $\theta = \beta$.
  - TASK: Find the area of polar region $D$.
  - Let **partition** $\mathcal{P} := \{\theta_0, \theta_1, \theta_2, \ldots, \theta_{N-1}, \theta_N\} \subset [\alpha, \beta]$ be **arbitrary**.
    Let **tags** $\mathcal{T} := \{\theta_1^*, \theta_2^*, \theta_3^*, \ldots, \theta_{N-1}^*, \theta_N^*\} \subset [\alpha, \beta]$ be **arbitrary**.
  - **Key element**: **Sector**

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|---|---|---|---|---|---|---|
| Angle | of | $k^{th}$ Sector | := | $\big($Length of $k^{th}$ subinterval$\big)$ | = | $\Delta\theta_k$ |
| Radius | of | $k^{th}$ Sector | := | (Distance from Pole to Curve) | = | $\lvert f(\theta_k^*)\rvert$ |
| Area | of | $k^{th}$ Sector | := | $\frac{1}{2} \times \Big(\text{Radius}\Big)^2 \times \Big(\text{Angle}\Big)$ | = | $\frac{1}{2}\lvert f(\theta_k^*)\rvert^2 \Delta\theta_k = \frac{1}{2}\Big[f(\theta_k^*)\Big]^2 \Delta\theta_k$ |

$$\therefore \quad \text{Area}(D) \approx A_N^* := \sum_{k=1}^{N} \frac{1}{2}\Big[f(\theta_k^*)\Big]^2 \Delta\theta_k \implies \text{Area}(D) = \lim_{N\to\infty} A_N^* = \boxed{\int_\alpha^\beta \frac{1}{2}\Big[f(\theta)\Big]^2 \, d\theta}$$

- **AREA OF RADIALLY SIMPLE POLAR REGION W/O POLE AS A BP: (FROM FIRST PRINCIPLES):**
  - SETUP: Given functions $f, g \in C[\alpha, \beta]$ s.t. $|f(\theta)| \geq |g(\theta)| \geq 0 \ \ \forall \theta \in [\alpha, \beta]$ and $0 < \beta - \alpha \leq 2\pi$.
    Let $D$ be the region bounded by the polar curves $r = f(\theta)$, $r = g(\theta)$ and rays $\theta = \alpha$, $\theta = \beta$.
  - TASK: Find the area of polar region $D$.
  - Let **partition** $\mathcal{P} := \{\theta_0, \theta_1, \theta_2, \ldots, \theta_{N-1}, \theta_N\} \subset [\alpha, \beta]$ be **arbitrary**.
    Let **tags** $\mathcal{T} := \{\theta_1^*, \theta_2^*, \theta_3^*, \ldots, \theta_{N-1}^*, \theta_N^*\} \subset [\alpha, \beta]$ be **arbitrary**.
  - **Key element**: **Polar Rectangle** (i.e. **Difference of Two Sectors**)

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|---|---|---|---|---|---|---|
| Angle | of | larger $k^{th}$ Sector | := | $\Big($Length of $k^{th}$ subinterval$\Big)$ | = | $\Delta\theta_k$ |
| Radius | of | larger $k^{th}$ Sector | := | $\Big($Distance from Pole to Curve $f\Big)$ | = | $\lvert f(\theta_k^*)\rvert$ |
| Area | of | larger $k^{th}$ Sector | := | $\frac{1}{2} \times \Big(\text{Radius}\Big)^2 \times \Big(\text{Angle}\Big)$ | = | $\frac{1}{2}\lvert f(\theta_k^*)\rvert^2 \Delta\theta_k = \frac{1}{2}\Big[f(\theta_k^*)\Big]^2 \Delta\theta_k$ |
| Angle | of | smaller $k^{th}$ Sector | := | $\Big($Length of $k^{th}$ subinterval$\Big)$ | = | $\Delta\theta_k$ |
| Radius | of | smaller $k^{th}$ Sector | := | $\Big($Distance from Pole to Curve $g\Big)$ | = | $\lvert g(\theta_k^*)\rvert$ |
| Area | of | smaller $k^{th}$ Sector | := | $\frac{1}{2} \times \Big(\text{Radius}\Big)^2 \times \Big(\text{Angle}\Big)$ | = | $\frac{1}{2}\lvert g(\theta_k^*)\rvert^2 \Delta\theta_k = \frac{1}{2}\Big[g(\theta_k^*)\Big]^2 \Delta\theta_k$ |
| Area | of | $k^{th}$ Polar Rectangle | := | $\Big(\text{Larger Sector}\Big) - \Big(\text{Smaller Sector}\Big)$ | = | $\frac{1}{2}\Big( [f(\theta_k^*)]^2 - [g(\theta_k^*)]^2 \Big)\Delta\theta_k$ |

$$\therefore \quad \text{Area}(D) \approx A_N^* := \sum_{k=1}^{N} \frac{1}{2}\Big( [f(\theta_k^*)]^2 - [g(\theta_k^*)]^2 \Big)\Delta\theta_k \implies \text{Area}(D) = \lim_{N\to\infty} A_N^* = \boxed{\int_\alpha^\beta \frac{1}{2}\left(\Big[f(\theta)\Big]^2 - \Big[g(\theta)\Big]^2\right) d\theta}$$

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- **AREA OF A RADIALLY SIMPLE ($r$-SIMPLE) POLAR REGION:**
  - Let $D$ be a $r$-simple region s.t. $D = \{(r, \theta) \in \mathbb{R}^2 : 0 \leq g_1(\theta) \leq r \leq g_2(\theta), \alpha \leq \theta \leq \beta \text{ s.t. } 0 < \beta - \alpha \leq 2\pi\}$. Then:

$$\text{Area}(D) = \frac{1}{2}\int_{\text{Smallest } \theta\text{-value in } D}^{\text{Largest } \theta\text{-value in } D} (\text{Outer BC})^2 - (\text{Inner BC})^2 \, d\theta = \frac{1}{2}\int_\alpha^\beta [g_2(\theta)]^2 - [g_1(\theta)]^2 \, d\theta$$

- **AREA OF A QUASI-RADIALLY SIMPLE (QUASI-$r$-SIMPLE) POLAR REGION:**
  - Let $D$ be a $r$-simple region s.t. $D = \{(r, \theta) \in \mathbb{R}^2 : 0 \leq g_1(\theta) \leq r \leq g_2(\theta), \alpha \leq \theta \leq \beta \text{ for } g_1(\theta), \gamma \leq \theta \leq \delta \text{ for } g_2(\theta)\}$.
    Moreover, let BP's $(g_1(\alpha), \alpha), (g_2(\gamma), \gamma)$ share one ray & BP's $(g_1(\beta), \beta), (g_2(\delta), \delta)$ share another ray. Then:

$$\text{Area}(D) = \frac{1}{2}\int_{\text{Smallest } \theta\text{-value for Outer BC}}^{\text{Largest } \theta\text{-value for Outer BC}} (\text{Outer BC})^2 \, d\theta - \frac{1}{2}\int_{\text{Smallest } \theta\text{-value for Inner BC}}^{\text{Largest } \theta\text{-value for Inner BC}} (\text{Inner BC})^2 \, d\theta$$

$$\implies \text{Area}(D) = \frac{1}{2}\int_\gamma^\delta [g_2(\theta)]^2 \, d\theta - \frac{1}{2}\int_\alpha^\beta [g_1(\theta)]^2 \, d\theta$$

- **HOW TO SUBDIVIDE A POLAR REGION THAT'S NEITHER $r$-SIMPLE NOR QUASI-$r$-SIMPLE:**

  Construct a ray that contains at least one BP that separates two outer BC's (or two inner BC's). Repeat as needed.

**EX 6.3.1:** Graph the polar curve $r = 2$.

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**EX 6.3.2:** Graph the polar curve $\theta = \frac{3\pi}{4}$.

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**EX 6.3.3:** (a) Convert the polar curve $r = 3\sec\theta$ to rectangular form. (b) Graph the curve.

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**EX 6.3.4:** Graph the polar curve $r = 4\theta$.

**EX 6.3.5:** Graph the circle $r = 3\cos\theta$.

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**EX 6.3.6:** Graph the cardioid $r = 3(1 + \sin\theta)$.

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**EX 6.3.7:** Graph the limaçon $r = 1 - 4\sin\theta$.

**EX 6.3.8:** Graph the rose $r = 4\sin(2\theta)$.

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**EX 6.3.9:** Graph the lemniscate $r^2 = 4\sin(2\theta)$.

**EX 6.3.10:** Let $R$ be the region enclosed by polar curves $r = 2\sin\theta$ and $r = 2\cos\theta$.

(a) Sketch the region $R$.

(b) Setup integral to find Area($R$).

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**EX 6.3.11:** Let $R$ be the region enclosed by one leaf of the rose $r = 4\cos(2\theta)$.

(a) Sketch the region $R$.

(b) Setup integral to find Area($R$).

**EX 6.3.12:** Let $R$ be the region outside the lemniscate $r^2 = 4\cos(2\theta)$ and inside the circle $r = 3\cos\theta$.

(a) Sketch the region $R$.

(b) Setup integral to find Area($R$).

**EX 6.3.13:** Compute the integral $I = \int_0^{\pi/8} \cos^2\theta \; d\theta.$

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**EX 6.3.14:** Compute the integral $I = \int_{\pi/4}^{\pi} \sin^2(2\theta) \; d\theta.$