ARC LENGTH & SURFACE AREA: [SST 6.4]

- **ARC LENGTH OF $y = f(x)$:**
  - SETUP: Given function $f \in C^1[a,b]$. Let Γ be the curve $y = f(x)$ bounded by the vertical lines $x = a$ & $x = b$.
  - TASK: Find the arc length of Γ
  - Let **partition** $\mathcal{P} := \{x_0, x_1, x_2, \ldots, x_{N-1}, x_N\}$ & **tags** $\mathcal{T} := \{x_1^*, x_2^*, \ldots, x_{N-1}^*, x_N^*\}$ s.t. $\mathcal{P} \subset [a,b]$
  - **Key element**: **Line Segment**
  - Length of k^{th} Line Segment $= \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2} = \sqrt{1 + \left(\frac{\Delta y_k}{\Delta x_k}\right)^2}\Delta x_k \overset{MVT}{=} \sqrt{1 + [f'(x_k^*)]^2}\Delta x_k$
  - $\text{ArcLength}(\Gamma) \approx L_N^* = \sum_{k=1}^{N} \sqrt{1 + [f'(x_k^*)]^2}\Delta x_k$
  - $\text{ArcLength}(\Gamma) = \lim_{N\to\infty} L_N^* = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$

- **ARC LENGTH OF $x = g(y)$:**
  - SETUP: Given function $g \in C^1[c,d]$. Let Γ be the curve $x = g(y)$ bounded by the horizontal lines $y = c$ & $y = d$.
  - TASK: Find the arc length of Γ
  - Let **partition** $\mathcal{P} := \{y_0, y_1, y_2, \ldots, y_{N-1}, y_N\}$ & **tags** $\mathcal{T} := \{y_1^*, y_2^*, \ldots, y_{N-1}^*, y_N^*\}$ s.t. $\mathcal{P} \subset [c,d]$
  - **Key element**: **Line Segment**
  - Length of k^{th} Line Segment $= \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2} = \sqrt{1 + \left(\frac{\Delta x_k}{\Delta y_k}\right)^2}\Delta y_k \overset{MVT}{=} \sqrt{1 + [g'(y_k^*)]^2}\Delta y_k$
  - $\text{ArcLength}(\Gamma) \approx L_N^* = \sum_{k=1}^{N} \sqrt{1 + [g'(y_k^*)]^2}\Delta y_k$
  - $\text{ArcLength}(\Gamma) = \lim_{N\to\infty} L_N^* = \int_c^d \sqrt{1 + [g'(y)]^2}\, dy$

- **ARC LENGTH OF POLAR CURVE $r = f(\theta)$:**
  - SETUP: Given function $f \in C^1[\alpha,\beta]$. Let Γ be the polar curve $r = f(\theta)$ bounded by the rays $\theta = \alpha$ & $\theta = \beta$.
  - TASK: Find the arc length of Γ
  - Instead of building the Riemann sum from scratch, just convert the key element's length into polar form:
  - Length of k^{th} Line Segment $= \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2} \overset{CI1}{=} \left(\frac{\Delta\theta_k}{\Delta\theta_k}\right)\sqrt{(\Delta x_k)^2 + (\Delta y_k)^2} = \sqrt{\left(\frac{\Delta x_k}{\Delta\theta_k}\right)^2 + \left(\frac{\Delta y_k}{\Delta\theta_k}\right)^2}\Delta\theta_k$
  - $\text{ArcLength}(\Gamma) \approx L_N^* = \sum_{k=1}^{N} \sqrt{\left(\frac{\Delta x_k}{\Delta\theta_k}\right)^2 + \left(\frac{\Delta y_k}{\Delta\theta_k}\right)^2}\Delta\theta_k$
  - $\text{ArcLength}(\Gamma) = \lim_{N\to\infty} L_N^* = \int_\alpha^\beta \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$
  - Now, the integral needs to be purely in terms of θ, so write $\frac{dx}{d\theta}$ & $\frac{dy}{d\theta}$ in polar coordinates:

$$\begin{cases} x = r\cos\theta = f(\theta)\cos\theta \\ y = r\sin\theta = f(\theta)\sin\theta \end{cases} \implies \begin{cases} \frac{dx}{d\theta} = f'(\theta)\cos\theta - f(\theta)\sin\theta \\ \frac{dy}{d\theta} = f'(\theta)\sin\theta + f(\theta)\cos\theta \end{cases} \implies \left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = [f(\theta)]^2 + [f'(\theta)]^2$$

  - $\text{ArcLength}(\Gamma) = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = \int_\alpha^\beta \sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\, d\theta$

- **SURFACE AREA OF CURVE $y = f(x)$ REVOLVED ABOUT THE X-AXIS**:
  - SETUP: Given $f \in C^1[a,b]$. Let Γ be the curve $y = f(x)$ bounded by the vertical lines $x = a$ & $x = b$.

    Let S be the surface formed by revolving Γ about the x-**axis**.
  - TASK: Find the surface area of S.
  - Let **partition** $\mathcal{P} := \{x_0, x_1, x_2, \ldots, x_{N-1}, x_N\}$ & **tags** $\mathcal{T} := \{x_1^*, x_2^*, \ldots, x_{N-1}^*, x_N^*\}$ s.t. $\mathcal{P} \subset [a,b]$
  - **Key element**: **V-Band**

Average Radius	of	k^{th} V-Band	:=	$\frac{1}{2}\left[f(x_{k-1}^*) + f(x_k^*)\right]$	$\approx$	$f(x_k^*)$
Slant Height	of	k^{th} V-Band	:=	$\sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}$	$\overset{MVT}{=}$	$\sqrt{1 + [f'(x_k^*)]^2}\,\Delta x_k$
Surface Area	of	k^{th} V-Band	:=	$2\pi \times \Big(\text{Average Radius}\Big) \times \Big(\text{Slant Height}\Big)$	$\approx$	$2\pi f(x_k^*)\sqrt{1 + [f'(x_k^*)]^2}\,\Delta x_k$

  - Riemann Sum: SurfaceArea(S) $\approx SA_N^* := \sum_{k=1}^{N} 2\pi f(x_k^*)\sqrt{1 + [f'(x_k^*)]^2}\,\Delta x_k$
  - Integral: SurfaceArea(S) $= \lim_{N\to\infty} SA_N^* = \boxed{\int_a^b 2\pi f(x)\sqrt{1 + [f'(x)]^2}\,dx}$

- **SURFACE AREA OF CURVE $x = g(y)$ REVOLVED ABOUT THE Y-AXIS**:
  - SETUP: Given $g \in C^1[c,d]$. Let Γ be the curve $x = g(y)$ bounded by the horizontal lines $y = c$ & $y = d$.

    Let S be the surface formed by revolving Γ about the y-**axis**.
  - TASK: Find the surface area of S.
  - Let **partition** $\mathcal{P} := \{y_0, y_1, y_2, \ldots, y_{N-1}, y_N\}$ & **tags** $\mathcal{T} := \{y_1^*, y_2^*, \ldots, y_{N-1}^*, y_N^*\}$ s.t. $\mathcal{P} \subset [c,d]$
  - **Key element**: **H-Band**

Average Radius	of	k^{th} H-Band	:=	$\frac{1}{2}\left[g(y_{k-1}^*) + g(y_k^*)\right]$	$\approx$	$g(y_k^*)$
Slant Height	of	k^{th} H-Band	:=	$\sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}$	$\overset{MVT}{=}$	$\sqrt{1 + [g'(y_k^*)]^2}\,\Delta y_k$
Surface Area	of	k^{th} H-Band	:=	$2\pi \times \Big(\text{Average Radius}\Big) \times \Big(\text{Slant Height}\Big)$	$\approx$	$2\pi g(y_k^*)\sqrt{1 + [g'(y_k^*)]^2}\,\Delta y_k$

  - Riemann Sum: SurfaceArea(S) $\approx SA_N^* := \sum_{k=1}^{N} 2\pi g(y_k^*)\sqrt{1 + [g'(y_k^*)]^2}\,\Delta y_k$
  - Integral: SurfaceArea(S) $= \lim_{N\to\infty} SA_N^* = \boxed{\int_c^d 2\pi g(y)\sqrt{1 + [g'(y)]^2}\,dy}$

- **SURFACE AREA IN POLAR COORDINATES**: Given $f \in C^1[\alpha,\beta]$.
  - Let Γ be the curve $r = f(\theta)$ bounded by rays $\theta = \alpha$ & $\theta = \beta$.
  - Let S be the surface formed by revolving Γ about the x-**axis**.
    - $$\boxed{\text{SurfaceArea(S)} = \int_\alpha^\beta 2\pi r\sin\theta\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta = \int_\alpha^\beta 2\pi f(\theta)\sin\theta\sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\,d\theta}$$
  - Let S be the surface formed by revolving Γ about the y-**axis**.
    - $$\boxed{\text{SurfaceArea(S)} = \int_\alpha^\beta 2\pi r\cos\theta\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta = \int_\alpha^\beta 2\pi f(\theta)\cos\theta\sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\,d\theta}$$

EX 6.4.1: Let Γ be the curve $y = x^3 + x - 1$ bounded by the lines $x = 1$ & $x = 4$.

Setup integral to find the arc length of Γ.

EX 6.4.2: Let Γ be the curve $x = \sqrt[5]{y}$ bounded by the lines $y = -3$ & $y = 7$.

Setup integral to find the arc length of Γ.

EX 6.4.3: Let Γ be the polar curve $r = 2\theta$ over the interval $\theta \in \left[0, \frac{3\pi}{4}\right]$.

(a) Setup integral to find the arc length of Γ.

(b) Setup integral to find the surface area of the surface S_1 formed by revolving Γ about the x-axis.

(c) Setup integral to find the surface area of the surface S_2 formed by revolving Γ about the y-axis.

EX 6.4.4: Let Γ be the curve $y = e^x$ over the interval $x \in [0, 2]$.

Let S be the surface formed by revolving Γ about the x-**axis**.

Setup integral to find the surface area of S.

EX 6.4.5: Let Γ be the curve $y = e^x$ over the interval $x \in [0, 2]$.

Let S be the surface formed by revolving Γ about the y-**axis**.

Setup integral to find the surface area of S.

EX 6.4.6: Let Γ be the curve $y = e^x$ over the interval $x \in [0, 2]$.

Let S be the surface formed by revolving Γ about the line $y = -2$.

Setup integral to find the surface area of S.

EX 6.4.7: Let Γ be the curve $y = e^x$ over the interval $x \in [0, 2]$.

Let S be the surface formed by revolving Γ about the line $y = 9$.

Setup integral to find the surface area of S.

EX 6.4.8: Let Γ be the curve $y = e^x$ over the interval $x \in [0, 2]$.

Let S be the surface formed by revolving Γ about the line $x = -3$.

Setup integral to find the surface area of S.

EX 6.4.9: Let Γ be the curve $y = e^x$ over the interval $x \in [0, 2]$.

Let S be the surface formed by revolving Γ about the line $x = 5$.

Setup integral to find the surface area of S.