WORK, FLUID FORCE, CENTROIDS: [SST 6.5]

- **ELEMENTARY DEFINITION OF WORK:**
  - The **work** done by a **constant force** on an object moving it a **constant distance** along a straight line:
  - $\Big(\text{Work}\Big) = \Big(\text{Force}\Big) \times \Big(\text{Distance}\Big)$

- **WORK DONE BY A VARIABLE FORCE ALONG THE X-AXIS:**
  - SETUP: Given function $F \in C[a,b]$ s.t. $y = F(x)$.
  - TASK: Find the work done by the variable force $F(x)$ in moving an object from $x = a$ to $x = b$
  - Let **partition** $\mathcal{P} := \{x_0, x_1, x_2, \ldots, x_{N-1}, x_N\} \subset [a,b]$ be **arbitrary**.

    Let **tags** $\mathcal{T} := \{x_1^*, x_2^*, x_3^*, \ldots, x_{N-1}^*, x_N^*\} \subset [a,b]$ be **arbitrary**.
  - **Key element**: **Subinterval** $[x_{k-1}, x_k]$

Distance	of	k^{th} Subinterval	$:=$	$\Big(\text{Length of } k^{th} \text{ subinterval}\Big)$	$=$	Δx_k
Force	along	k^{th} Subinterval		(Assuming Δx_k is small)	$\approx$	$F(x_k^*)$
Work	along	k^{th} Subinterval	$:=$	$\Big(\text{Force}\Big) \times \Big(\text{Distance}\Big)$	$=$	$F(x_k^*)\,\Delta x_k$

  - Work done by force $F(x)$ over $[a,b] \approx W_N^* := \displaystyle\sum_{k=1}^{N} F(x_k^*)\,\Delta x_k$
  - Work done by force $F(x)$ over $[a,b] = \displaystyle\lim_{N\to\infty} W_N^* = \boxed{\int_a^b F(x)\,dx}$

- **WORK DONE BY A VARIABLE FORCE ALONG THE Y-AXIS:**
  - SETUP: Given function $G \in C[c,d]$ s.t. $x = G(y)$.
  - TASK: Find the work done by the variable force $G(y)$ in moving an object from $y = c$ to $y = d$
  - Let **partition** $\mathcal{P} := \{y_0, y_1, y_2, \ldots, y_{N-1}, y_N\} \subset [c,d]$ be **arbitrary**.

    Let **tags** $\mathcal{T} := \{y_1^*, y_2^*, \ldots, y_{N-1}^*, y_N^*\} \subset [c,d]$ be **arbitrary**.
  - **Key element**: **Subinterval** $[y_{k-1}, y_k]$

Distance	of	k^{th} Subinterval	$:=$	$\Big(\text{Length of } k^{th} \text{ subinterval}\Big)$	$=$	Δy_k
Force	along	k^{th} Subinterval		(Assuming Δy_k is small)	$\approx$	$F(y_k^*)$
Work	along	k^{th} Subinterval	$:=$	$\Big(\text{Force}\Big) \times \Big(\text{Distance}\Big)$	$=$	$F(y_k^*)\,\Delta y_k$

  - Work done by force $G(y)$ over $[c,d] \approx W_N^* := \displaystyle\sum_{k=1}^{N} G(y_k^*)\,\Delta y_k$
  - Work done by force $G(y)$ over $[c,d] = \displaystyle\lim_{N\to\infty} W_N^* = \boxed{\int_c^d G(y)\,dy}$

- **HOOKE'S LAW FOR SPRINGS:**
  - SETUP: Given **spring** with **stiffness constant** $k > 0$ fixed at one end and freely move horizontally.
  - $x \equiv$ **distance** from the spring's **equilibrium position** (AKA **natural length**).
  - Hooke's Law: The **restoring force** $F(x) = kx$

- **REMARKS:**
  - For **physics problems**, it's crucial to choose a sensible **coordinate system**.
  - Be aware of the units of measure in work problems – conversions may be necessary:

Mass	Distance	Force	Work
kg	m	N	J
g	cm	dyne	erg
slug	ft	lb	ft-lb

EX 6.5.1: Find the work done if a (constant) force of 100 lb is used to push a rock along a line 50 ft.

EX 6.5.2: Find the work done if a force of 30 N is used to pull a block along the x-axis from $x = 5$ km to $x = 25$ km.

EX 6.5.3: Find the work done if a force of 20 dynes is used to push an object along a line 5 m.

EX 6.5.4: Find the work done in lifting a 15 lb bag of rice 4 ft by the handles.

EX 6.5.5: Find the work done in lifting a 2 ton straight horizontal beam 1000 ft. (1 ton = 2000 lb)

EX 6.5.6: A particle is moved along the x-axis by a force of $F(x) = x^4$ lbs at a point x ft from the origin.

(a) Find the work done in moving the particle from a position of $x = 2$ ft to $x = 4$ ft.

(b) Find the work done in moving the particle from the origin to a position of $x = 3$ ft.

EX 6.5.7: A spring has a natural length of 5 ft. Suppose a 10-lb force will compress it a length of 3 ft.

(a) Find the stiffness constant, k.

(b) How much work is required to stretch the spring 4 ft from its natural length?

(c) How much work is required to compress the spring 3 ft from its natural length?

EX 6.5.8:

(a) A cable that weighs 2 lb/ft is used to lift 900 lb cart of coal up a mine shaft 600 ft deep. Find the work done.

(b) Suppose halfway through the lifting process the very bottom piece of cable attached to the cart snaps.

How much work was done until the cable snapped?

- **WORK DONE PUMPING FLUID OUT OF A TANK:**

  - SETUP: **Tank** with **cross-sectional area** $A(y)$ filled with fluid of weight-density δ.
  - TASK: Find the work done pumping fluid out of tank.
  - Let **partition** $\mathcal{P} := \{y_0, y_1, y_2, \dots, y_{N-1}, y_N\}$ of fluid be **arbitrary**.

    Let **tags** $\mathcal{T} := \{y_1^*, y_2^*, y_3^*, \dots, y_{N-1}^*, y_N^*\}$ of fluid be **arbitrary**.
  - **Key element**: **H-Slab (of fluid)**

Weight-Density	of	Fluid	:=	(Weight per Volume of Fluid)	$\equiv$	δ
Thickness	of	k^{th} H-Slab	:=	(Length of k^{th} subinterval)	$=$	Δy_k
Distance to Spout	of	k^{th} H-Slab	:=	(y-coord. of spout) $-$ (y-coord. of H-Slab)	$\equiv$	$h(y_k^*)$
Area	of	k^{th} H-Slab		(Using appropiate geometric formula)	$\equiv$	$A(y_k^*)$
Volume	of	k^{th} H-Slab	:=	(Area) $\times$ (Thickness)	$=$	$A(y_k^*)\,\Delta y_k$
Weight	on	k^{th} H-Slab	:=	(Weight-Density) $\times$ (Volume)	$=$	$\delta A(y_k^*)\,\Delta y_k$
Work done	pumping	k^{th} H-Slab	:=	(Weight) $\times$ (Distance to Spout)	$=$	$\delta A(y_k^*)\,h(y_k^*)\,\Delta y_k$

  - Riemann Sum: Work done pumping fluid out of tank $\approx W_N^* := \sum_{k=1}^{N} \delta A(y_k^*)\,h(y_k^*)\,\Delta y_k$
  - Integral: Work done pumping fluid out of tank $= \lim_{N\to\infty} W_N^* = \boxed{\int_{\text{bottom } y\text{-coord. of fluid}}^{\text{top } y\text{-coord. of fluid}} \delta A(y)h(y)\;dy}$

- **REMARKS:**

  - For **physics problems**, it's crucial to choose a sensible **coordinate system**.

VARIABLE	PHYSICAL QUANTITY	DEFINITION	TYPICAL UNITS
g	Acceleration of Gravity	Distance per square-time	$\text{m/sec}^2, \text{ft/sec}^2, \text{mi/hr}^2$
ρ	Mass-Density	Mass per Volume	$\text{slugs/ft}^3, \text{kg/m}^3, \text{g/cm}^3$
δ	Weight-Density	Weight per Volume	$\text{lb/ft}^3, \text{N/m}^3, \text{dynes/cm}^3$
F	Fluid Force	Pressure $\times$ Area	lbs, N, dynes
W	Work	Force $\times$ Distance	ft-lb, J, ergs

$\star$ (Weight-Density) = (Mass-Density) $\times$ (Acceleration of Gravity) $\qquad \delta = \rho g$

EX 6.5.9: A 10 m tank has the shape of a circular cone with the base on the ground and base radius 4 m.

The tank is filled with water up to 7 m in height.

A 6 m vertical spout is attached to the top of the tank (which is the apex of the cone).

NOTE: The mass-density of water is $\rho = 1000 \text{ kg/m}^3$ & the acceleration of gravity is $g \approx 9.8 \text{ m/s}^2$.

(a) Sketch & characterize the tank & fluid. (Pick appropriate coordinate system)

(b) Setup integral to find the work required to empty the tank by pumping all the water out of the spout.

- **FLUID FORCE AGAINST A THIN PLATE SUBMERGED HORIZONTALLY:**
  - SETUP: **Thin plate** of area A is **submerged horizontally** at depth h in a fluid of weight-density δ.
  - $\boxed{\Big(\text{Fluid force}\Big) = \Big(\text{Pressure}\Big) \times \Big(\text{Area}\Big) \implies F = (\delta h)A}$
  - Fluid force does NOT depend on the shape or size of the container.

- **FLUID FORCE AGAINST A THIN PLATE SUBMERGED VERTICALLY:**
  - SETUP: **Thin plate submerged vertically** in a fluid of weight-density δ.
  - TASK: Find the fluid force against the thin plate.
  - Let **partition** $\mathcal{P} := \{y_0, y_1, y_2, \ldots, y_{N-1}, y_N\}$ of thin plate be **arbitrary**.

    Let **tags** $\mathcal{T} := \{y_1^*, y_2^*, y_3^*, \ldots, y_{N-1}^*, y_N^*\}$ of thin plate be **arbitrary**.
  - **Key element**: **H-Rect (of plate)**

Weight-Density	of	Fluid	:=	$\Big($Weight per Volume of Fluid$\Big)$	$\equiv$	δ
Width	of	k^{th} H-Rect	:=	$\Big($Length of k^{th} subinterval$\Big)$	$=$	Δy_k
Depth	of	k^{th} H-Rect	:=	$\Big(y$-coord. of Surface$\Big) - \Big(y$-coord. of H-Slab$\Big)$	$=$	$h\left(y_k^*\right)$
Length	of	k^{th} H-Rect	:=	$\Big($Right BC of Plate$\Big) - \Big($Left BC of Plate$\Big)$	$=$	$L\left(y_k^*\right)$
Pressure	on	k^{th} H-Rect	:=	$\Big($Weight-Density of Fluid$\Big) \times \Big($Depth$\Big)$	$=$	$\delta h\left(y_k^*\right)$
Area	of	k^{th} H-Rect	:=	$\Big($Length$\Big) \times \Big($Width$\Big)$	$=$	$L\left(y_k^*\right)\Delta y_k$
Fluid Force	of	k^{th} H-Rect	:=	$\Big($Pressure$\Big) \times \Big($Area$\Big)$	$=$	$\delta h\left(y_k^*\right) L\left(y_k^*\right)\Delta y_k$

  - Riemann Sum: Fluid force against thin plate $\approx F_N^* := \sum_{k=1}^{N} \delta h(y_k^*) L(y_k^*) \Delta y_k$
  - Integral: Fluid force against thin plate $= \lim_{N\to\infty} F_N^* = \boxed{\int_{\text{bottom } y\text{-coord. of plate}}^{\text{top } y\text{-coord. of plate}} \delta h(y) L(y)\ dy}$

- **FLUID FORCE AGAINST ONE (VERTICAL) END OF A RESERVOIR:**
  - SETUP: **Reservoir** is filled with fluid of weight-density δ.
  - TASK: Find the total fluid force against one end of the reservoir.
  - Let **partition** $\mathcal{P} := \{y_0, y_1, y_2, \ldots, y_{N-1}, y_N\}$ of fluid be **arbitrary**.

    Let **tags** $\mathcal{T} := \{y_1^*, y_2^*, y_3^*, \ldots, y_{N-1}^*, y_N^*\}$ of fluid be **arbitrary**.
  - **Key element**: **H-Rect (of fluid)**

Weight-Density	of	Fluid	:=	$\Big($Weight per Volume of Fluid$\Big)$	$\equiv$	δ
Width	of	k^{th} H-Rect	:=	$\Big($Length of k^{th} subinterval$\Big)$	$=$	Δy_k
Depth	of	k^{th} H-Rect	:=	$\Big(y$-coord. of Surface$\Big) - \Big(y$-coord. of H-Slab$\Big)$	$=$	$h\left(y_k^*\right)$
Length	of	k^{th} H-Rect	:=	$\Big($Right BC of Reservoir$\Big) - \Big($Left BC of Reservoir$\Big)$	$=$	$L\left(y_k^*\right)$
Pressure	on	k^{th} H-Rect	:=	$\Big($Weight-Density of Fluid$\Big) \times \Big($Depth$\Big)$	$=$	$\delta h\left(y_k^*\right)$
Area	of	k^{th} H-Rect	:=	$\Big($Length$\Big) \times \Big($Width$\Big)$	$=$	$L\left(y_k^*\right)\Delta y_k$
Fluid Force	of	k^{th} H-Rect	:=	$\Big($Pressure$\Big) \times \Big($Area$\Big)$	$=$	$\delta h\left(y_k^*\right) L\left(y_k^*\right)\Delta y_k$

  - Riemann Sum: Fluid force against end of reservoir $\approx F_N^* := \sum_{k=1}^{N} \delta h(y_k^*) L(y_k^*) \Delta y_k$
  - Integral: Fluid force against end of reservoir $= \lim_{N\to\infty} F_N^* = \boxed{\int_{\text{bottom } y\text{-coord. of fluid}}^{\text{top } y\text{-coord. of fluid}} \delta h(y) L(y)\ dy}$

- **REMARK:**

  ⋆ $\Big(\text{Weight-Density}\Big) = \Big(\text{Mass-Density}\Big) \times \Big(\text{Acceleration of Gravity}\Big) \qquad \delta = \rho g$

EX 6.5.10: A vertical plate is submerged in water and has the shape of an equilateral triangle with side length 6m orientated with a vertex at the top and a horizontal side at the bottom. The top of the plate is 2m below the water surface.

The Mass-Density of water $\rho = 1000$ kg/m^3. The Acceleration of Gravity $g \approx 9.8$ m/sec^2.

(a) Sketch & characterize the fluid & submerged vertical plate. (Pick appropriate coordinate system)

(b) Setup integral to find the fluid force against the vertical plate.

EX 6.5.11: Let both ends of a trough be vertical & shaped as the bottom-half of a circle of diameter 10 ft.

Suppose the trough is filled with gasoline 3 ft deep. $\left(\text{Weight-Density of gasoline } \delta = 42.0 \text{ lb/ft}^3\right)$

(a) Sketch & characterize the frontal view of one end of the trough. (Pick appropriate coordinate system)

(b) Setup integral to find the fluid force against one end of the trough.

- **CENTROIDS OF SIMPLE GEOMETRIC FIGURES:** $\left(\text{Mass-Density } \rho = 1\right)$

  - The centroid $(\bar{x}, \bar{y})$ of a **line segment** is its **midpoint**.
  - The centroid $(\bar{x}, \bar{y})$ of a **rectangle** or **circle** or **ellipse** is its **geometric center**.
  - the centroid $(\bar{x}, \bar{y}, \bar{z})$ of a **cube** or **sphere** or **ellipsoid** is its **geometric center**.

- **MOMENTS ABOUT THE X-AXIS & Y-AXIS OF V-ALIGNED THIN PLATE (LAMINA):**

  - SETUP: Given $f, g \in C[a,b]$ s.t. $f(x) \geq g(x) \quad \forall x \in [a,b]$ $\Big($i.e. curve f lies above curve g over $[a,b]\Big)$
    Let R be the region bounded by curves $y = f(x)$, $y = g(x)$, and the lines $x = a$, $x = b$.
    Lamina R is the thin plate with cross-section region R and uniform (constant) mass-density ρ.
  - TASK: Find the moments about the x-axis & y-axis of lamina R.
  - Let **partition** $\mathcal{P} := \{x_0, x_1, x_2, \ldots, x_{N-1}, x_N\} \subset [a,b]$ be **arbitrary**.
    Let **tags** $\mathcal{T} := \{x_1^*, x_2^*, x_3^*, \ldots, x_{N-1}^*, x_N^*\} \subset [a,b]$ be **arbitrary**.
  - **Key element**: **V-Rect**

Mass-Density	of	Lamina	:=	(Mass per Area of Lamina)	$\equiv$	ρ
Width	of	k^{th} V-Rect	:=	(Length of k^{th} subinterval)	$=$	Δx_k
Length	of	k^{th} V-Rect	:=	(Top BC) − (Bottom BC)	$=$	$f(x_k^*) - g(x_k^*)$
Centroid	on	k^{th} V-Rect	:=	(Geometric Center of V-Rect)	$\equiv$	$(\bar{x}_k, \bar{y}_k) = \left(x_k^*, \frac{1}{2}\left[f(x_k^*) + g(x_k^*)\right]\right)$
Mass	of	k^{th} V-Rect	:=	(Density) × (Area)	$\equiv$	$\Delta m_k = \rho\left[f(x_k^*) - g(x_k^*)\right]\Delta x_k$
Moment about y-axis	of	k^{th} V-Rect	:=	$\bar{x}_k \Delta m_k$	$=$	$\rho x_k^* \left[f(x_k^*) - g(x_k^*)\right]\Delta x_k$
Moment about x-axis	of	k^{th} V-Rect	:=	$\bar{y}_k \Delta m_k$	$=$	$\frac{1}{2}\rho\Big(\left[f(x_k^*)\right]^2 - \left[g(x_k^*)\right]^2\Big)\Delta x_k$

  - Riemann Sum: Mass of lamina $R \approx m^* := \sum_{k=1}^{N} \Delta m_k = \sum_{k=1}^{N} \rho\left[f(x_k^*) - g(x_k^*)\right]\Delta x_k$

    Riemann Sum: Moment about y-axis of lamina $R \approx M_y^* := \sum_{k=1}^{N} \bar{x}_k \Delta m_k = \sum_{k=1}^{N} \rho x_k^* \left[f(x_k^*) - g(x_k^*)\right]\Delta x_k$

    Riemann Sum: Moment about x-axis of lamina $R \approx M_x^* := \sum_{k=1}^{N} \bar{y}_k \Delta m_k = \sum_{k=1}^{N} \frac{1}{2}\rho\Big(\left[f(x_k^*)\right]^2 - \left[g(x_k^*)\right]^2\Big)\Delta x_k$

  - Integral: Mass of lamina $R \equiv m := \lim_{N\to\infty} m^* = \int_a^b \rho\left[f(x) - g(x)\right]\, dx$

    Integral: Moment about y-axis of lamina $R \equiv M_y := \lim_{N\to\infty} M_y^* = \int_a^b \rho x\left[f(x) - g(x)\right]\, dx$

    Integral: Moment about x-axis of lamina $R \equiv M_x := \lim_{N\to\infty} M_x^* = \int_a^b \frac{1}{2}\rho\Big(\left[f(x)\right]^2 - \left[g(x)\right]^2\Big)\, dx$

    Centroid of lamina R $\equiv (\bar{x}, \bar{y}) = \left(\dfrac{M_y}{m}, \dfrac{M_x}{m}\right)$

- **REMARKS:**

  - If a lamina's density $\rho \neq 1$, then its centroid is sometimes referred to as its **center of mass**.

EX 6.5.12: Let lamina R be the region bounded by the curves $y = \dfrac{1}{x}$ & $2x + 2y = 5$ and have uniform density $\rho > 0$.

(a) Sketch & characterize the lamina R.

(b) Setup integrals to compute: the mass of R, the moments about the x-axis & y-axis of R, the centroid of R.

EX 6.5.13: Let lamina R be the triangle with vertices (0,0), (0,1), and (3,2) and have uniform density $\rho > 0$.

(a) Sketch & characterize the lamina R.

(b) Setup integrals to compute: the mass of R, the moments about the x-axis & y-axis of R, the centroid of R.