INTEGRATION BY PARTS: [SST 7.2]

- **INTEGRATION BY PARTS (IBP):**
  - Recall: There's no general 'product rule' for integration.
  - Integration by Parts is the next best thing to a 'product rule' for integration.
  - SETUP: Let $u, v \in C^1[a,b]$ such that u, v are **elementary functions** of x. (i.e. not **piecewise**)
  - DERIVATION:
    - Recall the **product rule** for derivatives: $\frac{d}{dx}[uv] = v\frac{du}{dx} + u\frac{dv}{dx}$
    $\implies d[uv] = v\ du + u\ dv$
    $\implies \int d[uv] = \int v\ du + \int u\ dv$
    $\implies uv = \int v\ du + \int u\ dv$
  - INDEFINITE INTEGRAL FORM: $\int u\ dv = uv - \int v\ du$
  - DEFINITE INTEGRAL FORM: $\int_a^b u\ dv = \Big[uv\Big]_{x=a}^{x=b} - \int_a^b v\ du$
  - WHAT THE FORMULA MEANS:
    - Premise: $\int u\ dv$ is hard to integrate.
    - Change of variables: Pick function u & differential dv s.t. $\int v\ du$ is easy/easier to integrate.
    - Computation: Differentiate u to get du
    - Computation: Integrate dv to get v. Inserting **constant of integration** $\left(+C\right)$ is NOT necessary.
    - WARNING: Choose u & dv wisely! Bad choices can cause $\int v\ du$ to be harder or impossible to integrate!
    - REMARK: If the integral in question only involves **one function**, consider choosing $dv = 1\ dx$.

- **STRATEGY FOR CHOOSING u & dv WISELY:**
  - LIPTE Heuristic: Whichever function type comes first in the following list choose as u:

Letter	Function Type	Example Functions
L	Logarithms	$\ln x, \log y, \log_8 t, \ldots$
I	Inverse Trig	$\arcsin x, \arctan y, \text{arcsec}\ t, \ldots$
P	Polynomials	$x, y^2, 5t^3, \ldots$
T	Trig Functions	$\sin x, \tan\theta, \sec\omega, \ldots$
E	Exponentials	$e^x, 2^y, \left(\frac{1}{3}\right)^t, (-4)^x, \ldots$

  - If the integrand is a product of only **rational functions** and/or **roots**, IBP is likely not useful.
  - What about **compositions**??
    - DEFINITION: A function f is a **light composition** $\iff$ f has the form $f(ax+b)$, where $a, b \in \mathbb{R}$.
      - EXAMPLES: $\sin(\pi\theta - 4), \frac{1}{3t-2}, \sqrt{2z-1}, \arctan(x\sqrt{5}), e^{7y+3}, (9w-8)^4, \ldots$
    - DEFINITION: A function f is a **heavy composition** $\iff$ f is NOT a light composition.
      - EXAMPLES: $\sin(2\arccos x), \frac{1}{\ln x}, \sqrt{e^y}, \arctan(1/x), e^{\sqrt{t}}, \ln(\ln w), \ldots$
    - For **light compositions**, perform integration by parts as usual, afterwards u-substitution may be necessary.
    - For **heavy compositions**, consider an appropriate u-substitution first, afterwards do integration by parts.

- **TABULAR INTEGRATION:**
  - Is **more efficient** when **several iterations of integration by parts** are needed.
  - Works best when u is either a **polynomial**, **sine**, or **cosine**.

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EX 7.2.1: Evaluate $I = \int xe^x \; dx$.

EX 7.2.2: Evaluate $I = \int \omega \sin(2\omega) \; d\omega$.

EX 7.2.3: Evaluate $I = \int x^2 e^x \; dx$ **using integration by parts**.

EX 7.2.4: Evaluate $I = \int x^2 e^x \; dx$ **using tabular integration**.

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EX 7.2.5: Evaluate $I = \int e^{2x} \sin(3x)\, dx$ **using integration by parts**.

EX 7.2.6: Evaluate $I = \int e^{2x} \sin(3x)\, dx$ **using tabular integration**.

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EX 7.2.7: Evaluate $I = \int_0^1 x^4 e^{-2x} \, dx$.

EX 7.2.8: Evaluate $I = \int_1^e t \ln t \, dt$.

EX 7.2.9: Evaluate $I = \int \sin(\ln x) \, dx$.

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EX 7.2.10: Evaluate $I = \int \ln z \; dz$.

EX 7.2.11: Evaluate $I = \int_0^{1/\sqrt{2}} \arccos x \; dx$.

EX 7.2.12: Evaluate $I = \int \frac{\ln x}{x^2} \; dx$.

EX 7.2.13: Evaluate $I = \int_1^4 \sqrt{p} \ln p \; dp$.

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EX 7.2.14: Derive the **reduction formula**: $\int x^n e^x \, dx = x^n e^x - n \int x^{n-1} e^x \, dx$, where $n \in \mathbb{N}$.

EX 7.2.15: Derive the **reduction formula**: $\int (\ln t)^n \, dt = t (\ln t)^n - n \int (\ln t)^{n-1} \, dt$, where $n \in \mathbb{N}$.

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