INTEGRATION: TRIGOMETRIC INTEGRALS [SST 7.3]

- $\int \sin^m x \cos^n x \, dx$ $\quad \Big($Here, $m, n \in \bar{\mathbb{N}} := \{0, 1, 2, 3, \cdots\}\Big)$
  - (CASE I) m is **odd** $\quad \Longrightarrow m = 2k+1$, where $k \in \bar{\mathbb{N}}$:
    * <u>RELEVANT TRIG IDENTITY:</u> $\sin^2 x + \cos^2 x = 1$
    * <u>RELEVANT INTEGRAL RULE:</u> $\int u^n \, du = \frac{1}{n+1} u^{n+1} + C$ $\quad \Big($**Power Rule**$\Big)$
    * $\int \sin^{2k+1} x \cos^n x \, dx = \int \left(\sin^2 x\right)^k \cos^n x \sin x \, dx = \int \left(1 - \cos^2 x\right)^k \cos^n x \sin x \, dx$
    * Change of variables (CV): Let $u = \cos x \Longrightarrow du = -\sin x \, dx \Longrightarrow \sin x \, dx = -du$
    * $\Longrightarrow \int \sin^{2k+1} x \cos^n x \, dx = \int \left(1 - \cos^2 x\right)^k \cos^n x \sin x \, dx \overset{CV}{=} \int \left(1 - u^2\right)^k u^n (-du)$
    * Use the **Binomial Theorem**, **Pascal's Triangle**, and **Power Rule** as needed.
  - (CASE II) n is **odd** $\quad \Longrightarrow n = 2k+1$, where $k \in \bar{\mathbb{N}}$:
    * <u>RELEVANT TRIG IDENTITY:</u> $\sin^2 x + \cos^2 x = 1$
    * <u>RELEVANT INTEGRAL RULE:</u> $\int u^n \, du = \frac{1}{n+1} u^{n+1} + C$ $\quad \Big($**Power Rule**$\Big)$
    * $\int \sin^m x \cos^{2k+1} x \, dx = \int \sin^m x \left(\cos^2 x\right)^k \cos x \, dx = \int \sin^m x \left(1 - \sin^2 x\right)^k \cos x \, dx$
    * Change of variables (CV): Let $u = \sin x \Longrightarrow du = \cos x \, dx \Longrightarrow \cos x \, dx = du$
    * $\Longrightarrow \int \sin^m x \cos^{2k+1} x \, dx = \int \sin^m x \left(1 - \sin^2 x\right)^k \cos x \, dx \overset{CV}{=} \int u^m \left(1 - u^2\right)^k \, du$
    * Use the **Binomial Theorem**, **Pascal's Triangle**, and **Power Rule** as needed.
  - (CASE III) m is **even** AND $n = 0$ $\quad \Longrightarrow m = 2k$, where $k \in \mathbb{N} := \{1, 2, 3, \cdots\}$:
    * <u>RELEVANT TRIG IDENTITIES:</u> $\sin^2 x = \frac{1}{2}(1 - \cos(2x))$, $\cos^2 x = \frac{1}{2}(1 + \cos(2x))$
    * <u>RELEVANT INTEGRAL RULE:</u> $\int \cos(ax) \, dx = \frac{1}{a} \sin(ax) + C$
    * $\int \sin^{2k} x \, dx = \int \left(\sin^2 x\right)^k \, dx = \int \left[\frac{1}{2}(1 - \cos(2x))\right]^k \, dx$
    * Use the **Binomial Theorem**, **Pascal's Triangle**, and the identity for $\cos^2 x$ as needed.
  - (CASE IV) $m = 0$ AND n is **even** $\quad \Longrightarrow n = 2k$, where $k \in \mathbb{N}$:
    * <u>RELEVANT TRIG IDENTITIES:</u> $\sin^2 x = \frac{1}{2}(1 - \cos(2x))$, $\cos^2 x = \frac{1}{2}(1 + \cos(2x))$
    * <u>RELEVANT INTEGRAL RULE:</u> $\int \cos(ax) \, dx = \frac{1}{a} \sin(ax) + C$
    * $\int \cos^{2k} x \, dx = \int \left(\cos^2 x\right)^k \, dx = \int \left[\frac{1}{2}(1 + \cos(2x))\right]^k \, dx$
    * Use the **Binomial Theorem**, **Pascal's Triangle**, and the identity for $\cos^2 x$ as needed.
  - (CASE X) m, n are **both even such that** $m \geq 4$ **and** $n \geq 4$. **IGNORE THIS CASE**
    * This case requires the use of **half-angle** & **product-to-sum identities**, making this case extremely tedious!
    * Thus, this case will NOT be considered in this course.

- **Binomial Theorem:** NOTE: $\binom{n}{k} := \frac{n!}{k!(n-k)!}$ and $n! := (n)(n-1)(n-2)\cdots(3)(2)(1)$

  $$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^n = \binom{n}{0} x^n y^0 + \binom{n}{1} x^{n-1} y^1 + \binom{n}{2} x^{n-2} y^2 + \cdots + \binom{n}{n-1} x^1 y^{n-1} + \binom{n}{n} x^0 y^n$$

- **Pascal's Triangle:**

```
                1
             1     1
          1     2     1
       1     3     3     1
    1     4     6     4     1
 1     5    10    10     5     1
```

EX 7.3.1: Evaluate $I = \int \sin^6 x \cos^5 x \; dx$.

EX 7.3.2: Evaluate $I = \int \sin^5(3\theta) \; d\theta$.

EX 7.3.3: Evaluate $I = \int \cos^4(2\omega) \; d\omega$.

EX 7.3.4: Evaluate $I = \int \sin^6 t \; dt$.

INTEGRATION: TRIGOMETRIC INTEGRALS [SST 7.3]

- $\int \tan^m x \sec^n x \; dx$ $\quad \Big(\text{Here, } m,n \in \bar{\mathbb{N}} := \{0,1,2,3,\cdots\}\Big)$
  - (**CASE I**) n is **even** $\quad \Longrightarrow \; n = 2k$, where $k \in \mathbb{N} := \{1,2,3,\cdots\}$
    - RELEVANT TRIG IDENTITY: $1 + \tan^2 x = \sec^2 x$
    - RELEVANT INTEGRAL RULE: $\int u^n \; du = \frac{1}{n+1} u^{n+1} + C$ $\quad$ (**Power Rule**)
    - $\int \tan^m x \sec^{2k} x \; dx = \int \tan^m x \left(\sec^2 x\right)^{k-1} \sec^2 x \; dx = \int \tan^m x \left(1 + \tan^2 x\right)^{k-1} \sec^2 x \; dx$
    - Change of variables (CV): Let $u = \tan x \implies du = \sec^2 x \; dx \implies \sec^2 x \; dx = du$
    - $\implies \int \tan^m x \sec^{2k} x \; dx = \int \tan^m x \left(1 + \tan^2 x\right)^{k-1} \sec^2 x \; dx \overset{CV}{=} \int u^m \left(1 + u^2\right)^{k-1} \; du$
    - Use the **Binomial Theorem**, **Pascal's Triangle**, and **Power Rule** as needed.
  - (**CASE II**) m is **odd** AND $n \neq 0$ $\quad \Longrightarrow \; m = 2k+1$, where $k \in \bar{\mathbb{N}}$
    - RELEVANT TRIG IDENTITY: $1 + \tan^2 x = \sec^2 x$
    - RELEVANT INTEGRAL RULE: $\int u^n \; du = \frac{1}{n+1} u^{n+1} + C$ $\quad$ (**Power Rule**)
    - $\int \tan^{2k+1} x \sec^n x \; dx = \int \left(\tan^2 x\right)^k \sec^{n-1} x \sec x \tan x \; dx = \int \left(\sec^2 x - 1\right)^k \sec^{n-1} x \sec x \tan x \; dx$
    - Change of variables (CV): Let $u = \sec x \implies du = \sec x \tan x \; dx \implies \sec x \tan x \; dx = du$
    - $\implies \int \tan^{2k+1} x \sec^n x \; dx = \int \left(\sec^2 x - 1\right)^k \sec^{n-1} x \sec x \tan x \; dx \overset{CV}{=} \int \left(u^2 - 1\right)^k u^{n-1} \; du$
    - Use the **Binomial Theorem**, **Pascal's Triangle**, and **Power Rule** as needed.
  - (**CASE X**) All other cases. $\quad$ (CV ≡ "**C**hange of **V**ariables", IBP ≡ "**I**ntegration **B**y **P**arts")
    - Requires clever use of **trig identities**, **CV**, **IBP**, $\int \tan x \; dx$, and/or $\int \sec x \; dx = \ln|\sec x + \tan x| + C$.
    - For **large powers** ($n \geq 5$), use **reduction formula**: $\int \sec^n(\alpha u) \; du = \frac{\sec^{n-2}(\alpha u)\tan(\alpha u)}{\alpha(n-1)} + \frac{n-2}{n-1} \int \sec^{n-2}(\alpha u) \; du$

- $\int \cot^m x \csc^n x \; dx$ $\quad \Big(\text{Here, } m,n \in \bar{\mathbb{N}} := \{0,1,2,3,\cdots\}\Big)$
  - (**CASE I**) n is **even** $\quad \Longrightarrow \; n = 2k$, where $k \in \mathbb{N}$
    - RELEVANT TRIG IDENTITY: $1 + \cot^2 x = \csc^2 x$
    - RELEVANT INTEGRAL RULE: $\int u^n \; du = \frac{1}{n+1} u^{n+1} + C$ $\quad$ (**Power Rule**)
    - $\int \cot^m x \csc^{2k} x \; dx = \int \cot^m x \left(\csc^2 x\right)^{k-1} \csc^2 x \; dx = \int \cot^m x \left(1 + \cot^2 x\right)^{k-1} \csc^2 x \; dx$
    - Change of variables (CV): Let $u = \cot x \implies du = -\csc^2 x \; dx \implies \csc^2 x \; dx = -du$
    - $\implies \int \cot^m x \csc^{2k} x \; dx = \int \cot^m x \left(1 + \cot^2 x\right)^{k-1} \csc^2 x \; dx \overset{CV}{=} \int u^m \left(1 + u^2\right)^{k-1} \; (-du)$
  - (**CASE II**) m is **odd** and $n \neq 0$ $\quad \Longrightarrow \; m = 2k+1$, where $k \in \bar{\mathbb{N}}$
    - RELEVANT TRIG IDENTITY: $1 + \cot^2 x = \csc^2 x$
    - RELEVANT INTEGRAL RULE: $\int u^n \; du = \frac{1}{n+1} u^{n+1} + C$ $\quad$ (**Power Rule**)
    - $\int \cot^{2k+1} x \csc^n x \; dx = \int \left(\cot^2 x\right)^k \csc^{n-1} x \csc x \cot x \; dx = \int \left(\csc^2 x - 1\right)^k \csc^{n-1} x \csc x \cot x \; dx$
    - Change of variables (CV): Let $u = \csc x \implies du = -\csc x \cot x \; dx \implies \csc x \cot x \; dx = -du$
    - $\implies \int \cot^{2k+1} x \csc^n x \; dx = \int \left(\csc^2 x - 1\right)^k \csc^{n-1} x \csc x \cot x \; dx \overset{CV}{=} \int \left(u^2 - 1\right)^k u^{n-1} \; (-du)$
  - (**CASE X**) All other cases. $\quad$ (CV ≡ "**C**hange of **V**ariables", IBP ≡ "**I**ntegration **B**y **P**arts")
    - Requires clever use of **trig identities**, **CV**, **IBP**, $\int \cot x \; dx$, and/or $\int \csc x \; dx = -\ln|\csc x + \cot x| + C$.
    - For **large powers** ($n \geq 5$), use **reduction formula**: $\int \csc^n(\alpha u) \; du = -\frac{\csc^{n-2}(\alpha u)\cot(\alpha u)}{\alpha(n-1)} + \frac{n-2}{n-1} \int \csc^{n-2}(\alpha u) \; du$

EX 7.3.5: Evaluate $I = \int \tan^4 x \sec^6 x \; dx$.

EX 7.3.6: Evaluate $I = \int \tan^3(5\theta) \sec^8(5\theta) \; d\theta$.

EX 7.3.7: Evaluate $I = \int \tan^3 t \; dt$.

EX 7.3.8: Evaluate $I = \int \sec^3 \omega \; d\omega$.

INTEGRATION: TRIGOMETRIC INTEGRALS [SST 7.3]

$\Big(\text{Here, } m, n \in \mathbb{Z} \setminus \{0\}\Big)$

- $\displaystyle\int \sin(mx)\cos(nx)\ dx$
  - RELEVANT TRIG IDENTITIES: $\left\{\begin{array}{l} \sin(A+B) = \sin A\cos B + \cos A\sin B \\ \sin(A-B) = \sin A\cos B - \cos A\sin B \end{array}\right\}$

    $\implies 2\sin A\cos B = \sin(A+B) + \sin(A-B) \implies \sin A\cos B = \frac{1}{2}\left[\sin(A+B) + \sin(A-B)\right]$
  - RELEVANT INTEGRAL RULE: $\displaystyle\int \sin(ax)\ dx = -\frac{1}{a}\cos(ax) + C$
  - $\displaystyle\implies \int \sin(mx)\cos(nx)\ dx = \frac{1}{2}\int \sin[(m+n)x]\ dx + \frac{1}{2}\int \sin[(m-n)x]\ dx$

- $\displaystyle\int \cos(mx)\cos(nx)\ dx$
  - RELEVANT TRIG IDENTITIES: $\left\{\begin{array}{l} \cos(A+B) = \cos A\cos B - \sin A\sin B \\ \cos(A-B) = \cos A\cos B + \sin A\sin B \end{array}\right\}$

    $\implies 2\cos A\cos B = \cos(A+B) + \cos(A-B) \implies \cos A\cos B = \frac{1}{2}\left[\cos(A+B) + \cos(A-B)\right]$
  - RELEVANT INTEGRAL RULE: $\displaystyle\int \cos(ax)\ dx = \frac{1}{a}\sin(ax) + C$
  - $\displaystyle\implies \int \cos(mx)\cos(nx)\ dx = \frac{1}{2}\int \cos[(m+n)x]\ dx + \frac{1}{2}\int \cos[(m-n)x]\ dx$

- $\displaystyle\int \sin(mx)\sin(nx)\ dx$
  - RELEVANT TRIG IDENTITIES: $\left\{\begin{array}{l} \cos(A+B) = \cos A\cos B - \sin A\sin B \\ \cos(A-B) = \cos A\cos B + \sin A\sin B \end{array}\right\}$

    $\implies -2\sin A\sin B = \cos(A+B) - \cos(A-B) \implies \sin A\sin B = -\frac{1}{2}\left[\cos(A+B) - \cos(A-B)\right]$
  - RELEVANT INTEGRAL RULE: $\displaystyle\int \cos(ax)\ dx = \frac{1}{a}\sin(ax) + C$
  - $\displaystyle\implies \int \sin(mx)\sin(nx)\ dx = -\frac{1}{2}\int \cos[(m+n)x]\ dx + \frac{1}{2}\int \cos[(m-n)x]\ dx$

- **Negative Angle Identities**:
  - ⋆ $\sin(-x) = -\sin x$
  - ⋆ $\cos(-x) = \cos(x)$

EX 7.3.9: Evaluate $I = \int \sin(13x)\cos(7x)\ dx$.

EX 7.3.10: Evaluate $I = \int \cos(-13\theta)\cos(7\theta)\ d\theta$.

EX 7.3.11: Evaluate $I = \int \sin(13t)\sin(-7t)\ dt$.

EX 7.3.12: Evaluate $I_{m,n} = \displaystyle\int_{-\pi}^{\pi} \sin(mx)\cos(nx)\ dx$, where $m, n \in \mathbb{N}$. $\Big($HINT: Consider the cases $m = n$ & $m \neq n\Big)$

EX 7.3.13: Evaluate $I_{m,n} = \displaystyle\int_{-\pi}^{\pi} \cos(mx)\cos(nx)\ dx$, where $m, n \in \mathbb{N}$. $\Big($HINT: Consider the cases $m = n$ & $m \neq n\Big)$

EX 7.3.14: Evaluate $I_{m,n} = \displaystyle\int_{-\pi}^{\pi} \sin(mx)\sin(nx)\ dx$, where $m, n \in \mathbb{N}$. $\Big($HINT: Consider the cases $m = n$ & $m \neq n\Big)$

INTEGRATION: TRIGOMETRIC SUBSTITUTION [SST 7.3]

$\Big(\text{Here, } a > 0 \text{ and } k \in \bar{\mathbb{N}} := \{0, 1, 2, \cdots\}\Big)$

- $\displaystyle\int \sqrt{a^2 - u^2}\, du,\ \int u^{2k+1}\sqrt{a^2 - u^2}\, du,\ \int \frac{1}{\sqrt{a^2 - u^2}}\, du,\ \int \frac{u^{2k+1}}{\sqrt{a^2 - u^2}}\, du,\ \int \frac{1}{u^{2k}\sqrt{a^2 - u^2}}\, du,\ \int \frac{\sqrt{a^2 - u^2}}{u^{2k}}\, du$
  - CV: Let $\left[u = a\sin\theta \iff \sin\theta = \frac{u}{a} \iff \theta = \arcsin\left(\frac{u}{a}\right)\right] \implies du = a\cos\theta\, d\theta$ and $\sqrt{a^2 - u^2} = a\cos\theta$
  - Evaluate resulting integral using earlier techniques.
  - Build the appropriate **reference triangle** in order to perform **back substitution**.
  - <u>REMARK:</u> In order for arcsine to exist, $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \equiv \overline{QI} \cup \overline{QIV} \equiv \overline{RHP}$ (**Closed Right-Half Plane**)

- $\displaystyle\int \sqrt{a^2 + u^2}\, du,\ \int u^{2k+1}\sqrt{a^2 + u^2}\, du,\ \int \frac{1}{\sqrt{a^2 + u^2}}\, du,\ \int \frac{u^{2k+1}}{\sqrt{a^2 + u^2}}\, du,\ \int \frac{1}{u^{2k}\sqrt{a^2 + u^2}}\, du,\ \int \frac{\sqrt{a^2 + u^2}}{u^{2k+4}}\, du$
  - CV: Let $\left[u = a\tan\theta \iff \tan\theta = \frac{u}{a} \iff \theta = \arctan\left(\frac{u}{a}\right)\right] \implies du = a\sec^2\theta\, d\theta$ and $\sqrt{a^2 + u^2} = a\sec\theta$
  - Evaluate resulting integral using earlier techniques.
  - Build the appropriate **reference triangle** in order to perform **back substitution**.
  - <u>REMARK:</u> In order for arctangent to exist, $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \equiv QI \cup QIV \equiv RHP$ (**Open Right-Half Plane**)

- $\displaystyle\int \sqrt{u^2 - a^2}\, du,\ \int u^{2k+1}\sqrt{u^2 - a^2}\, du,\ \int \frac{1}{\sqrt{u^2 - a^2}}\, du,\ \int \frac{u^{2k+1}}{\sqrt{u^2 - a^2}}\, du,\ \int \frac{1}{u^{2k}\sqrt{u^2 - a^2}}\, du,\ \int \frac{\sqrt{u^2 - a^2}}{u^{2k}}\, du$
  - CV: Let $\left[u = a\sec\theta \iff \sec\theta = \frac{u}{a} \iff \theta = \text{arcsec}\left(\frac{u}{a}\right)\right] \implies du = a\sec\theta\tan\theta\, d\theta$ and $\sqrt{u^2 - a^2} = \pm a\tan\theta$
  - Evaluate resulting integral using earlier techniques.
  - Build the appropriate **reference triangle** in order to perform **back substitution**.
  - <u>REMARK:</u> In order for arcsecant to exist, $\theta \in [0, \pi] \setminus \left\{\frac{\pi}{2}\right\} \equiv \overline{QI} \cup \overline{QII} \equiv \overline{THP}$ (**Closed Top-Half Plane**)
    - * If $u > a > 0$, then $\sec\theta = \frac{u}{a} = \frac{(+)}{(+)} > 0 \implies \theta \in QI \implies \tan\theta > 0 \implies \sqrt{u^2 - a^2} = a\tan\theta$.
    - * If $u < -a < 0$, then $\sec\theta = \frac{u}{a} = \frac{(-)}{(+)} < 0 \implies \theta \in QII \implies \tan\theta < 0 \implies \sqrt{u^2 - a^2} = -a\tan\theta$.

- Forms involving $\sqrt{ax^2 + bx + c}$, where $a \neq 0$, $b \neq 0$, and $c \in \mathbb{R}$
  - First, **complete the square**: (CI-0 means "**C**lever **I**nsertion of **Zero**")
    - * e.g. $x^2 + 5x + 8 \overset{CI-0}{=} x^2 + 5x + 8 + \left[\left(\frac{5}{2}\right)^2 - \left(\frac{5}{2}\right)^2\right] = \left[x^2 + 5x + \left(\frac{5}{2}\right)^2\right] + 8 - \left(\frac{5}{2}\right)^2 = \left(x + \frac{5}{2}\right)^2 + \left(\frac{\sqrt{7}}{2}\right)^2$
    - * e.g. $3x^2 - 5x - 8 = 3\left(x^2 - \frac{5}{3}x - \frac{8}{3}\right) \overset{CI-0}{=} 3\left[x^2 - \frac{5}{3}x + \left(\frac{5}{6}\right)^2\right] + 3\left[-\frac{8}{3} - \left(\frac{5}{6}\right)^2\right] = 3\left[\left(x - \frac{5}{6}\right)^2 - \left(\frac{11}{6}\right)^2\right]$
    - * e.g. $2 - 2x - x^2 = -\left(x^2 + 2x - 2\right) \overset{CI-0}{=} -\left[x^2 + 2x + (1 - 1) - 2\right] = -\left[(x + 1)^2 - 3\right] = \left(\sqrt{3}\right)^2 - (x + 1)^2$
  - Next, perform a **change of variables**:
    - * e.g. $\sqrt{x^2 + 5x + 8} = \sqrt{\left(x + \frac{5}{2}\right)^2 + \left(\frac{\sqrt{7}}{2}\right)^2} = \sqrt{u^2 + a^2}$ where $u = x + \frac{5}{2}$ and $a = \frac{\sqrt{7}}{2}$.
    - * e.g. $\sqrt{3x^2 - 5x - 8} = \sqrt{3\left[\left(x - \frac{5}{6}\right)^2 - \left(\frac{11}{6}\right)^2\right]} = \sqrt{3}\sqrt{u^2 - a^2}$ where $u = x - \frac{5}{6}$ and $a = \frac{11}{6}$.
    - * e.g. $\sqrt{2 - 2x - x^2} = \sqrt{\left(\sqrt{3}\right)^2 - (x + 1)^2} = \sqrt{a^2 - u^2}$ where $u = x + 1$ and $a = \sqrt{3}$.
  - Finally, the integral is now a form involving: $\sqrt{u^2 - a^2}$, $\sqrt{u^2 + a^2}$, or $\sqrt{a^2 - u^2}$.

    Apply appropriate method above, then **back-substitute**.

- $\displaystyle\int \frac{1}{ax^2 \pm bx + c}\, dx$, where $a \neq 0$, $b \neq 0$, and $c \in \mathbb{R}$
  - First, **complete the square**, then perform a **change of variables**, finally apply appropriate method above.

EX 7.3.15: Evaluate: (a) $\int \frac{1}{\sqrt{1-x^2}}\,dx$ (b) $\int x^3\sqrt{1-x^2}\,dx$ (c) $\int \sqrt{1-x^2}\,dx$

First, pick appropriate trig substitution & sketch reference triangle:

EX 7.3.16: Evaluate: (a) $\int \frac{x^3}{\sqrt{4-x^2}}\,dx$ (b) $\int \frac{1}{x^2\sqrt{4-x^2}}\,dx$ (c) $\int \frac{\sqrt{4-x^2}}{x^2}\,dx$.

First, pick appropriate trig substitution & sketch reference triangle:

EX 7.3.17: Evaluate: (a) $\displaystyle\int \frac{1}{\sqrt{4+y^2}}\,dy$ (b) $\displaystyle\int y^3\sqrt{4+y^2}\,dy$ (c) $\displaystyle\int \sqrt{4+y^2}\,dy$

First, pick appropriate trig substitution & sketch reference triangle:

EX 7.3.18: Evaluate: (a) $\displaystyle\int \frac{y^3}{\sqrt{9+y^2}}\,dy$ (b) $\displaystyle\int \frac{1}{y^2\sqrt{9+y^2}}\,dy$ (c) $\displaystyle\int \frac{\sqrt{9+y^2}}{y^4}\,dy.$

First, pick appropriate trig substitution & sketch reference triangle:

EX 7.3.19: Evaluate: (a) $\displaystyle\int \frac{1}{\sqrt{t^2-9}}\,dt$ $\boxed{\text{(b)}}$ $\displaystyle\int t^3\sqrt{t^2-9}\,dt$ (c) $\displaystyle\int \sqrt{t^2-9}\,dt$

First, pick appropriate trig substitution & sketch reference triangle:

EX 7.3.20: Evaluate: (a) $\displaystyle\int \frac{t^3}{\sqrt{t^2-3}}\,dt$ $\boxed{\text{(b)}}$ $\displaystyle\int \frac{1}{t^2\sqrt{t^2-3}}\,dt$ (c) $\displaystyle\int \frac{\sqrt{t^2-3}}{t^2}\,dt.$

First, pick appropriate trig substitution & sketch reference triangle:

EX 7.3.21: Evaluate: (a) $\displaystyle\int \frac{1}{\sqrt{2x^2-8x+6}}\,dx$ (b) $\displaystyle\int \frac{1}{2x^2-8x+6}\,dx$ (c) $\displaystyle\int \frac{1}{x^2+2x+8}\,dx.$

EX 7.3.22: Evaluate $I = \displaystyle\int_0^{\sqrt{5}/2} \frac{1}{\sqrt{5-r^2}}\,dr.$