INTEGRATION: PARTIAL FRACTIONS [SST 7.4]

- **REVIEW OF FACTORING POLYNOMIALS:**
  - Monomial Factoring: e.g. $8x^6 + 2x^4 = 2x^4\left(4x^2+1\right)$
  - Factoring by Grouping: e.g. $x^3+3x^2+9x+27 = x^2(x+3)+9(x+3) = \left(x^2+9\right)(x+3)$
  - Difference of Squares: $A^2 - B^2 = (A+B)(A-B)$
  - Difference of Cubes: $A^3 - B^3 = (A-B)(A^2+AB+B^2)$
  - Sum of Cubes: $A^3 + B^3 = (A+B)(A^2-AB+B^2)$
  - REMARK: In this section, all 5^{th}-degree or higher polynomials will be factored a priori.

- **PARTIAL FRACTION DECOMPOSITION (PFD):**
  - TASK: Perform **PFD** on **rational function** $\dfrac{N(x)}{D(x)}$ where N & D are **polynomials**.
  - Recall: The **degree** of a polynomial is the **power** of its **highest-power term**.
    - e.g. $\deg\left[4x^3-7x^2+6x-100\right]=3$, $\deg\left[2-x^5-x^8\right]=8$, $\deg[5x]=1$, $\deg[17]=0$
  - STEP 1: If $\deg[N(x)] \geq \deg[D(x)]$, then perform **polynomial division**:
    - $\dfrac{N(x)}{D(x)} = Q(x) + \dfrac{R(x)}{D(x)}$, where Q & R are polynomials s.t. $\deg[R(x)] < \deg[D(x)]$.
  - STEP 2: Completely factor the **denominator** $D(x)$:
    - **Linear Factors** $(px+q)^m$ where $m \in \mathbb{N}$ and $p, q \in \mathbb{R}$ s.t. $p \neq 0$.
    - **Irreducible Quadratics** $\left(ax^2+bx^2+c\right)^n$ where $n \in \mathbb{N}$ and $a, b, c \in \mathbb{R}$ s.t $a \neq 0$.
      - Irreducible quadratics cannot be factored into linear factors with **real coefficients** $\iff b^2-4ac<0$.
      - Examples of irreducible quadratics: x^2+x+1, x^2+1
      - Examples of **reducible quadratics**: $x^2-1=(x+1)(x-1), x^2+2x+1=(x+1)^2, x^2-2x-3=(x-3)(x+1)$
    - For each **linear factor** $(px+q)^m$, the PFD must include the sum:
      $\dfrac{A_1}{(px+q)} + \dfrac{A_2}{(px+q)^2} + \cdots + \dfrac{A_m}{(px+q)^m}$, where $A_1, A_2, \ldots, A_m \in \mathbb{R}$
    - For each **irreducible quadratic** $\left(ax^2+bx^2+c\right)^n$, the PFD must include the sum:
      $\dfrac{B_1x+C_1}{ax^2+bx+c} + \dfrac{B_2x+C_2}{\left(ax^2+bx+c\right)^2} + \cdots + \dfrac{B_nx+C_n}{\left(ax^2+bx+c\right)^n}$, where $B_1, B_2, \ldots, B_n, C_1, C_2, \ldots, C_n \in \mathbb{R}$
  - STEP 3: In order to find the values of all the A_k's, B_k's, and/or C_k's, one must **setup & solve a linear system**.
  - STEP 4: The resulting integrals are now simple to work with.
    - For some integrals, you may need to **change variables** and/or **complete the square**.

  - IMPORTANT REMARKS ABOUT FACTORING A GENERAL n^{th}-DEGREE POLYNOMIAL:
    - **Fundamental Theorem of Algebra (FTA):** Every n^{th}-degree **polynomial with complex coefficients** can be factored into n **linear factors with complex coefficients**, some of which may be repeated.
    - **Corollary to FTA:** Every n^{th}-degree **polynomial with real coefficients** can be factored into **linears & irreducible quadratics with real coefficients**, some of which may be repeated.
    - **REMARK:** The FTA & this corollary merely asserts **existence of such factorizations**. They provide **no systematic procedure to actually find such factorizations**.
    - Here are some seemingly simple-looking polynomials that are hard to factor into linears & irreducible quadratics (you won't see these on the exams, so don't memorize them):
      - e.g. $x^4+1=\left(x^2+\sqrt{2}x+1\right)\left(x^2-\sqrt{2}x+1\right)$
      - e.g. $x^5+1=(x+1)\left(x^2-\frac{1+\sqrt{5}}{2}x+1\right)\left(x^2-\frac{1-\sqrt{5}}{2}x+1\right)$
      - e.g. $x^5-1=(x-1)\left(x^2+\frac{1+\sqrt{5}}{2}x+1\right)\left(x^2+\frac{1-\sqrt{5}}{2}x+1\right)$
      - e.g. $x^6+1=\left(x^2+1\right)\left(x^2+\sqrt{3}x+1\right)\left(x^2-\sqrt{3}x+1\right)$
      - e.g. $x^6-1=(x+1)(x-1)\left(x^2+x+1\right)\left(x^2-x+1\right)$

EX 7.4.1: Evaluate $I = \int \frac{x^3 + 2x^2 + x + 1}{x^2 + 6x}\, dx$.

EX 7.4.2: Evaluate $I = \int \frac{1}{x^2 + x - 2}\, dx$.

EX 7.4.3: Evaluate $I = \int \frac{3 - t}{t^2 + 2t + 1}\, dt$.

EX 7.4.4: Evaluate $I = \displaystyle\int \frac{1}{x(x-1)(6x-3)}\, dx.$

EX 7.4.5: Evaluate $I = \displaystyle\int \frac{1}{(x^2+1)(x^2+2)}\, dx.$

EX 7.4.6: Evaluate $I = \displaystyle\int \frac{x+2}{x(x^2-x-1)}\, dx.$

EX 7.4.7: Evaluate $I = \int \frac{\cos\theta}{\sin^2\theta + \sin\theta}\, d\theta$.

EX 7.4.8: Evaluate $I = \int \frac{1}{1-\sqrt{x}}\, dx$.

EX 7.4.9 Evaluate $I = \int \frac{1}{\sqrt{x} - \sqrt[3]{x}}\, dx$.