IMPROPER INTEGRALS [SST 7.7]

THE FUNDAMENTAL THEOREM OF CALCULUS (FTC):

- Recall the 1^{st} **FTC**: $f \in C[a,b] \implies \int_a^b f(x)\,dx = F(b) - F(a)$ s.t. $F'(x) = f(x) \ \ \forall x \in [a,b]$.
- Notice that $[a,b]$ is a **bounded interval** & f must be **continuous** on the interval.
- The FTC can be "extended" to **unbounded intervals** and/or functions with **break discontinuities**.

INTEGRATION OVER UNBOUNDED INTERVALS:

- **Unbounded intervals**: $[a,\infty), (-\infty,b], \mathbb{R} := (-\infty,\infty)$
- $f \in C[a,\infty) \implies \int_a^\infty f(x)\,dx = \lim_{B\to\infty} \int_a^B f(x)\,dx$
- $f \in C(-\infty,b] \implies \int_{-\infty}^b f(x)\,dx = \lim_{A\to-\infty} \int_A^b f(x)\,dx$
- $f \in C(\mathbb{R}) \implies \int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^c f(x)\,dx + \int_c^\infty f(x)\,dx$, where $c \in \mathbb{R}$
- REMARK: The **improper integral converges** if the corresponding **limit exists and is finite**. Otherwise, it **diverges**.
- **WARNING:** In general, $\int_{-\infty}^\infty f(x)\,dx \neq \lim_{L\to\infty} \int_{-L}^L f(x)\,dx$, e.g. $\int_{-\infty}^\infty \frac{1+x}{1+x^2}\,dx$ diverges, but $\lim_{L\to\infty} \int_{-L}^L \frac{1+x}{1+x^2}\,dx = \pi$.

INTEGRATION OF FUNCTIONS WITH BREAK DISCONTINUITIES:

- RECALL: f has a **break discontinuity** at $x = c$ if at least one 1-sided limit is infinite:
  - ⋆ $\left[\lim_{x\to c^-} f(x) = -\infty \text{ or } \infty\right]$ AND/OR $\left[\lim_{x\to c^+} f(x) = -\infty \text{ or } \infty\right]$
- Suppose $f \in C[a,b)$ s.t. f has a **break discontinuity** at $x = b$, then $\int_a^b f(x)\,dx = \lim_{B\to b^-} \int_a^B f(x)\,dx$
- Suppose $f \in C(a,b]$ s.t. f has a **break discontinuity** at $x = a$, then $\int_a^b f(x)\,dx = \lim_{A\to a^+} \int_A^b f(x)\,dx$
- Suppose $f \in C[a,b]$ s.t. f has a **break discontinuity** at $x = c \in (a,b)$, then $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$
- REMARK: The **improper integral converges** if the corresponding **limit exists and is finite**. Otherwise, it **diverges**.

INTUITIVE PROPERTIES OF INFINITY:

- Remember, ∞ is **not a real number**, but rather a **symbol** indicating **growth without bound**.
- Similarly, $-\infty$ indicates **decay without bound**.
- However, even though $\pm\infty$ are symbols, they satisfy some arithmetic properties that agree with intuition:
  - (E.1) $\infty + \infty = \infty \qquad -\infty - \infty = -\infty$
  - (E.2) $\forall x \in \mathbb{R}, \ \ \infty + x = x + \infty = \infty$ and $-\infty + x = x - \infty = -\infty$
  - (E.3) $(\infty)(\infty) = \infty, \qquad (-\infty)(-\infty) = \infty, \qquad (-\infty)(\infty) = (\infty)(-\infty) = -\infty$
  - (E.4) $x > 0 \implies x \cdot \infty = \infty$ and $x \cdot (-\infty) = -\infty, \qquad x < 0 \implies x \cdot \infty = -\infty$ and $x \cdot (-\infty) = \infty$
  - (E.5) $n \in \mathbb{N} \implies \infty^n = \infty$ and $\sqrt[n]{\infty} = \infty$

L'HÔPITAL'S RULE:

- Let $f, g \in C^1(a,b)$ and $c \in (a,b)$ s.t. $g'(x) \neq 0 \ \forall x \in (a,b)$ except possibly at c.
  If $\lim_{x\to c} \frac{f(x)}{g(x)} \overset{NS}{=} \frac{0}{0}$ OR $\pm\frac{\infty}{\infty}$, then $\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}$
- NOTATION: "NS" means "**N**aive **S**ubstitution":
  - e.g. $\lim_{x\to 2} x^3 \overset{NS}{=} (2)^3 = 8, \quad \lim_{x\to\infty} \sqrt{x+5} \overset{NS}{=} \sqrt{(\infty)+5} \overset{E.2}{=} \sqrt{\infty} \overset{E.5}{=} \infty$
  - SEE [LIMITS] LECTURE NOTES FOR MORE INFORMATION & PRACTICE.
- **INDETERMINANT FORMS:** $\frac{0}{0}, \frac{\infty}{\infty}, 0\cdot\infty, \infty - \infty, 0^0, \infty^0, 1^\infty$ $\Big(\infty$ can be replaced with $-\infty$, except $\infty - \infty\Big)$

EX 7.7.1: Evaluate $I = \int_1^\infty \frac{1}{x^2}\,dx$.

EX 7.7.2: Evaluate $I = \int_1^\infty \frac{1}{x}\,dx$.

EX 7.7.3: Evaluate $I = \int_{-\infty}^0 \frac{1}{\sqrt{1-x}}\,dx$.

EX 7.7.4: Evaluate $I = \int_{-\infty}^\infty \frac{1}{1+x^2}\,dx$.

EX 7.7.5: Evaluate $I = \int_0^1 \frac{1}{\sqrt{x}}\, dx$.

EX 7.7.6: Evaluate $I = \int_1^2 \frac{1}{(x-2)^4}\, dx$.

EX 7.7.7: Evaluate $I = \int_{-2}^0 \frac{1}{(x+1)^{2/3}}\, dx$.

EX 7.7.8: Evaluate $I = \int_0^{\infty} \frac{1}{2\sqrt{x}(x+1)}\,dx$.

EX 7.7.9: Consider the **Gamma Function** $\Gamma(\alpha) = \int_0^{\infty} x^{\alpha-1}e^{-x}\,dx$, where $\alpha \in \mathbb{R}_+ := (0,\infty)$.

The Gamma Function is encountered in many areas such as physics, fluid dynamics, probability theory & number theory.

(a) Find $\Gamma(1)$ (b) Find $\Gamma(2)$ (c) Show that $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$