SEQUENCES [SST 8.1]

DEFINITIONS:

- So far in Calculus, a typical **function** f has **domain** $\text{Dom}(f) \subseteq \mathbb{R}$ and **range** $\text{Rng}(f) \subseteq \mathbb{R}$.
- A **sequence** $\{a_n\}_{n=1}^{\infty}$ is a function s.t. $\text{Dom}(\{a_n\}) = \mathbb{N} := \{1, 2, 3, 4, 5, \cdots\}$ and $\text{Rng}(\{a_n\}) \subset \mathbb{R}$.
- NOTATION: Sequence is denoted either $\{a_n\}_{n=1}^{\infty}$ or $\{a_n\}$ or $(a_1, a_2, a_3, a_4, a_5, \cdots)$, where the **index** $n \in \mathbb{N}$.
- A sequence $\{a_n\}_{n=1}^{\infty}$ has the limit L, denoted $\lim\limits_{n\to\infty} a_n = L$, if successive terms approach L as n increases without bound.
- If $\lim\limits_{n\to\infty} a_n$ exists, we say the sequence **converges**. Otherwise, the sequence **diverges**.

LIMIT LAWS FOR SEQUENCES: Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be **convergent sequences** and $k \in \mathbb{R}$. Then:

(SQ.0) $\lim\limits_{n\to\infty} k = k$

(SQ.1) $\lim\limits_{n\to\infty} ka_n = k \lim\limits_{n\to\infty} a_n$

(SQ.2) $\lim\limits_{n\to\infty} (a_n \pm b_n) = \lim\limits_{n\to\infty} a_n \pm \lim\limits_{n\to\infty}$

(SQ.3) $\lim\limits_{n\to\infty} a_n b_n = \lim\limits_{n\to\infty} a_n \cdot \lim\limits_{n\to\infty} b_n$

(SQ.4) $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \dfrac{\lim\limits_{n\to\infty} a_n}{\lim\limits_{n\to\infty} b_n}$, **provided** $\lim\limits_{n\to\infty} b_n \neq 0$

(SQ.5) $\lim\limits_{n\to\infty} a_n^p = \left[\lim\limits_{n\to\infty} a_n\right]^p$, **provided** $p > 0$ and $a_n > 0 \quad \forall n \in \mathbb{N}$

CONTINUOUS FUNCTION THEOREM FOR SEQUENCES:

- Given sequence $\{a_n\}_{n=1}^{\infty}$ s.t. $\lim\limits_{n\to\infty} a_n = L \in \mathbb{R}$ and function f **continuous at** L and defined at all a_n.

  Then, $\lim\limits_{n\to\infty} f(a_n) = f\left(\lim\limits_{n\to\infty} a_n\right) = f(L)$.

GENERATING CURVE THEOREM (GCT):

- Given sequence $\{a_n\}_{n=1}^{\infty}$ and function $f \in C[1, \infty)$ s.t. $a_n = f(n) \quad \forall n \in \mathbb{N}$.

  Then, $\lim\limits_{x\to\infty} f(x) = L \in \mathbb{R} \implies \lim\limits_{n\to\infty} a_n = L$.
- WARNING: **The converse is not true in general!** That is, $\lim\limits_{n\to\infty} a_n = L \;\not\Longrightarrow\; \lim\limits_{x\to\infty} f(x) = L \in \mathbb{R}$, in general.

  The GCT works best when the sequence contains no trig fcn's & no factorials.

SQUEEZE THEOREM FOR SEQUENCES (SQZ):

- Given sequences $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$, $\{c_n\}_{n=1}^{\infty}$ s.t. $a_n \leq b_n \leq c_n \quad \forall n \geq N$ for some index N.

  Then, $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} c_n = L \in \mathbb{R} \implies \lim\limits_{n\to\infty} b_n = L$

L'HÔPITAL'S RULE (LHOP):

- Let $f, g \in C^1(a, b)$ and $c \in (a, b)$ s.t. $g'(x) \neq 0 \; \forall x \in (a, b)$ except possibly at c.

  If $\lim\limits_{x\to c} \dfrac{f(x)}{g(x)} \overset{NS}{=} \dfrac{0}{0}$ OR $\pm\dfrac{\infty}{\infty}$, then $\lim\limits_{x\to c} \dfrac{f(x)}{g(x)} = \lim\limits_{x\to c} \dfrac{f'(x)}{g'(x)}$
- **INDETERMINANT FORMS:** $\dfrac{0}{0}, \dfrac{\infty}{\infty}, 0 \cdot \infty, \infty - \infty, 0^0, \infty^0, 1^\infty$ $\Big(\infty$ can be replaced with $-\infty$, except $\infty - \infty\Big)$

MONOTONICITY & BOUNDEDNESS OF SEQUENCES:

- Sequence $\{a_n\}$ is **increasing** if $a_n \leq a_{n+1} \quad \forall n \in \mathbb{N}$. That is, $a_1 \leq a_2 \leq a_3 \leq a_4 \leq \cdots$ e.g. $\{n^3\} := (1, 8, 27, 64, 125, \cdots)$
- Sequence $\{a_n\}$ is **decreasing** if $a_n \geq a_{n+1} \quad \forall n \in \mathbb{N}$. That is, $a_1 \geq a_2 \geq a_3 \geq a_4 \geq \cdots$ e.g. $\left\{\frac{1}{n^2}\right\} = \left(1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \frac{1}{25}, \cdots\right)$
- Sequence $\{a_n\}$ is **monotone** if it's either increasing or decreasing.
- Sequence $\{a_n\}$ is **bounded above** if $\exists M \in \mathbb{R}$ s.t. $a_n \leq M \quad \forall n \in \mathbb{N}$. e.g. $\left\{\frac{1}{n}\right\}$ with $M = 1$.
- Sequence $\{a_n\}$ is **bounded below** if $\exists m \in \mathbb{R}$ s.t. $a_n \geq m \quad \forall n \in \mathbb{N}$. e.g. $\left\{\frac{1}{n}\right\}$ with $m = 0$.
- Sequence $\{a_n\}$ is **bounded** if it's both bounded above and bounded below. e.g. $\{\arctan n\}$ with $M = \pi/2$ and $m = 0$.

BOUNDED MONTONE CONVERGENCE THEOREM (BMCT): Every bounded monotone sequence converges.

- BCMT is mainly used to determine convergence of **recursive sequences**.
- Determining monotonicity & boundedness of **recursive sequences** requires rigorous analysis which is covered in an Advanced Calculus course (MATH 4350).

EX 8.1.1: List the first five terms of the sequence $\left\{\frac{2n+1}{n+3}\right\}_{n=1}^{\infty}$.

EX 8.1.2: List the first five terms of the sequence $\{a_n\}_{n=1}^{\infty}$, where $a_n = (-1)^{n+1}\left(\frac{n}{n+1}\right)$.

EX 8.1.3: List the first five terms of the **recursive sequence** $\{F_n\}_{n=1}^{\infty}$ s.t. $\begin{cases} F_{n+2} = F_n + F_{n+1} \\ F_1 = 4 \\ F_2 = -3 \end{cases}$

EX 8.1.4: Find the limit of sequence $\left\{2 - \frac{3}{n+4}\right\}_{n=1}^{\infty}$. Does the sequence converge?

EX 8.1.5: Find the limit of sequence $\{a_n\}$ where $a_n = \frac{1+n}{4+\sqrt{n}}$. Does the sequence converge?

EX 8.1.6: Find the limit of sequence $\{(-1)^n\}$. Does the sequence converge?

EX 8.1.7: Find the limit of sequence $\left\{\sqrt{\dfrac{9n-1}{n+2}}\right\}_{n=1}^{\infty}$. Does the sequence converge?

EX 8.1.8: Find the limit of sequence $\{\sqrt[n]{5}\}_{n=1}^{\infty}$. Does the sequence converge?

EX 8.1.9: Find the limit of sequence $\{\sqrt[n]{n}\}_{n=1}^{\infty}$. Does the sequence converge?

EX 8.1.10: Find the limit of sequence $\left\{\dfrac{n!}{n^n}\right\}_{n=1}^{\infty}$. Does the sequence converge?

EX 8.1.11: Find the limit of sequence $\left\{ \dfrac{(-1)^n \sin^7\left(0.7n^3\pi\right)}{n^2} \right\}_{n=1}^{\infty}$. Does the sequence converge?

EX 8.1.12: Find the limit of sequence $\left\{ \cos\left(\dfrac{\pi}{n}\right) \right\}_{n=1}^{\infty}$. Does the sequence converge?

EX 8.1.13: Find the limit of sequence $\{\sin(n\pi)\}_{n=1}^{\infty}$. Does the sequence converge?

EX 8.1.14: Find the limit of sequence $\left\{ \sin\left(\dfrac{n\pi}{2}\right) \right\}_{n=1}^{\infty}$. Does the sequence converge?

EX 8.1.15: Find the limit of sequence $\left\{ \left(1 + \dfrac{2}{n}\right)^{3n} \right\}_{n=1}^{\infty}$. Does the sequence converge?

EX 8.1.16: (a) Is the sequence $\{1-n\}_{n=1}^{\infty}$ increasing? decreasing? bounded above? bounded below? (b) Does it converge?

EX 8.1.17: (a) Is the sequence $\left\{(-1)^n-(-1)^{n+1}\right\}_{n=1}^{\infty}$ monotone? bounded? (b) Does it converge?