INFINITE SERIES: GEOMETRIC SERIES & TELESCOPING SERIES [SST 8.2]

DEFINITIONS: Let $\{a_n\}_{n=N}^{\infty}$ (where $N \in \mathbb{Z}$) be a **sequence**. That is, $\text{Dom}(\{a_n\}) \subset \mathbb{Z}$ and $\text{Rng}(\{a_n\}) \subset \mathbb{R}$.

- The n^{th} **partial sum** is defined by: $S_n = \sum_{k=1}^{n} a_k$. e.g. The 5^{th} partial sum is $S_5 = a_1 + a_2 + a_3 + a_4 + a_5$.
- An **infinite series** $\sum_{k=N}^{\infty} a_k = a_N + a_{N+1} + a_{N+2} + \cdots$ is a **sequence of partial sums** $\{S_n\}_{n=N}^{\infty}$ where $S_n := \sum_{k=1}^{n} a_k$.
  - The RHS of $\sum_{k=N}^{\infty} a_k = a_N + a_{N+1} + a_{N+2} + \cdots$ is called the **series expansion**. The LHS is called the **closed form**.
- k is called the **index** of the series $\sum_{k=N}^{\infty} a_k$.
  - Any index can be used: $\sum_{i=N}^{\infty} a_i = \sum_{j=N}^{\infty} a_j = \sum_{k=N}^{\infty} a_k = \sum_{n=N}^{\infty} a_n = \sum_{\alpha=N}^{\infty} a_\alpha = \sum_{\omega=N}^{\infty} a_\omega$.
  - Similarly with definite integrals: $\int_a^b f(x)\, dx = \int_a^b f(y)\, dy = \int_a^b f(z)\, dz = \int_a^b f(\alpha)\, d\alpha = \int_a^b f(\omega)\, d\omega$
- We'll never consider **finite series**, $\sum_{k=N}^{M} a_k$, as Calculus is not necessary in computing or analysing them.
  - Thus going forward, any mention of just "series" means "infinite series".

SERIES CONVERGENCE IS RELATED TO SEQUENCE CONVERGENCE:

- **Series** $\sum_{k=N}^{\infty} a_k$ **converges** $\iff$ Sequence of partial sums $\{S_n\}_{n=N}^{\infty}$ converges. Otherwise, the series **diverges**.

GENERAL PROPERTIES OF SERIES: $\Big(c, d \in \mathbb{R}\Big)$

- (SR.L) $\sum_{k=N}^{\infty} a_k$ and $\sum_{k=N}^{\infty} b_k$ both converge $\implies \sum_{k=N}^{\infty} (ca_k + db_k)$ converges $\implies \sum_{k=N}^{\infty} (ca_k + db_k) = c\sum_{k=N}^{\infty} a_k + d\sum_{k=N}^{\infty} b_k$
- Either $\sum_{k=N}^{\infty} a_k$ or $\sum_{k=N}^{\infty} b_k$ diverges $\implies \sum_{k=N}^{\infty} (ca_k + db_k)$ diverges

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ARITHMETIC SERIES: $\Big(a, b \in \mathbb{R}\Big)$

- (SR.A) **Arithmetic series** $\sum_{k=0}^{\infty} (ak + b)$ converges only when $a = b = 0$, otherwise it diverges.

GEOMETRIC SERIES:

- **Geometric series** has the form: $\sum_{k=0}^{\infty} ar^k = a + ar + ar^2 + ar^3 + ar^4 + ar^5 + \cdots$ where $a \neq 0$.
- (SR.G) $|r| < 1 \implies \sum_{k=0}^{\infty} ar^k$ converges $\implies \sum_{k=0}^{\infty} ar^k = \dfrac{a}{1-r}$ Otherwise, $|r| \geq 1 \implies \sum_{k=0}^{\infty} ar^k$ diverges $\Big(a \neq 0\Big)$
- Note that the **index** k must start at **zero**. If it doesn't, **re-index:**
  - (IDX) e.g. $\sum_{k=-2}^{\infty} ar^k = ar^{-2} + ar^{-1} + \sum_{k=0}^{\infty} ar^k$, e.g. $\sum_{k=3}^{\infty} ar^k \overset{CI-0}{=} -ar^0 - ar^1 - ar^2 + \sum_{k=0}^{\infty} ar^k$

TELESCOPING SERIES:

- (SR.T) A series is called a **telescoping series** if there is **internal cancellation of terms in the partial sums**.
- Basic Forms: $\sum_{k=N}^{\infty} [f(k) - f(k+1)]$ or $\sum_{k=N}^{\infty} [f(k) - f(k+1)]$, where f is a **function**.
- Useful tools: Partial Fraction Decomposition (PFD), Properties of Logarithms, Rationalizing Roots, Trig Identities.

EX 8.2.1: Expand the first five terms of each series:

(a) $\sum_{k=-2}^{\infty} k^5 =$

(b) $\sum_{j=1}^{\infty} \frac{1}{j^2} =$

(c) $\sum_{n=0}^{\infty} (-1)^n \sin\left(\frac{n\pi}{2}\right) =$

EX 8.2.2: Write each series expansion in closed form (NOTE: Closed forms are not unique):

(a) $1^2 + 2^2 + 3^2 + 4^2 + 5^2 + \cdots =$

(b) $(-1) + 1 + 8 + 27 + 64 + \cdots =$

(c) $7 - \sqrt{7} + 1 - \frac{1}{\sqrt{7}} + \frac{1}{7} + \cdots =$

EX 8.2.3: (a) Find the n^{th} partial sum, S_n, of the series $\sum_{k=1}^{\infty}(k+100)$. (b) Does the series converge?

EX 8.2.4: (a) Find the n^{th} partial sum, S_n, of the series $\sum_{k=1}^{\infty}(-1)^k$. (b) Does the series converge?

EX 8.2.5: Does the series $\sum_{k=0}^{\infty} \frac{11}{(-3)^{k+2}}$ converge? If so, what's its sum?

EX 8.2.6: Does the series $\sum_{k=0}^{\infty} \frac{(-4)^k}{(-5)^{2k+1}}$ converge? If so, what's its sum?

EX 8.2.7 Does the series $\sum_{k=-3}^{\infty} \frac{1}{6^{k+1}}$ converge? If so, what's its sum?

EX 8.2.8 Does the series $\left(\frac{3}{2}\right)^4 + \left(\frac{3}{2}\right)^5 + \left(\frac{3}{2}\right)^6 + \left(\frac{3}{2}\right)^7 + \left(\frac{3}{2}\right)^8 + \cdots$ converge? If so, what's its sum?

EX 8.2.9 Does the series $\sum_{k=0}^{\infty} \left[7\left(\frac{1}{2}\right)^k - 5\left(\frac{2}{3}\right)^k\right]$ converge? If so, what's its sum?

EX 8.2.10 Does the series $\sum_{k=1}^{\infty} \frac{1+3^k}{2^k}$ converge? If so, what's its sum?

EX 8.2.11 Does the series $\sum_{k=5}^{\infty} \frac{1}{k(k+1)}$ converge? If so, what's its sum?

EX 8.2.12 Does the series $\sum_{k=1}^{\infty} \left(e^{-k} - e^{-(k+1)}\right)$ converge? If so, what's its sum?

EX 8.2.13 Does the series $\sum_{k=1}^{\infty} \ln\left(1 + \frac{1}{k}\right)$ converge? If so, what's its sum?