POSITIVE SERIES: INTEGRAL TEST, p-SERIES [SST 8.3]

THE SAD TRUTH ABOUT COMPUTING THE SUM OF A CONVERGENT SERIES:

- So far, we've seen two types of series whose sum can be determined (if convergent): geometric series & telescoping series.
- In general, it's very hard or impossible to determine the sum of a series by hand.
- Going forward, the focus will be on **determining convergence using a collection of convergence tests**.
- Later, we will find sums of certain series using **Taylor series** [SST 8.8].
- In higher math courses, **Complex Analysis** and **Fourier Analysis** can be used to sum certain series.

MORE SERIES NOTATION:

- In instances where the starting index doesn't matter, the series will be denoted by $\sum a_k$.
- This notation will be mostly used in the **statement of theorems & convergence tests**.

INSERTING/REMOVING FINITELY MANY TERMS DOES NOT ALTER CONVERGENCE OR DIVERGENCE:

- e.g. $\sum_{k=0}^{\infty} a_k$ converges (diverges) $\implies \sum_{k=8}^{\infty} a_k$ converges (diverges) $\implies \sum_{k=-17}^{\infty} a_k$ converges (diverges).
- e.g. WARNING: The inserted terms must be defined: e.g. $\sum_{k=-3}^{\infty} \frac{1}{k}$ is not well-defined since the term $a_0 = \frac{1}{0}$ is undefined.

POSITIVE SERIES: $\sum a_k$ is called a **positive series** if each term $a_k \geq 0 \;\; \forall k$.

DIVERGENCE TEST: $\lim_{k\to\infty} a_k \neq 0 \implies \sum a_k$ diverges.

- TRANSLATION: "If the terms of the series do NOT converge to zero, then the series diverges."
- MEANING: Suppose $\lim_{k\to\infty} a_k = \frac{1}{3}$. Then, "eventually" the series $\sum a_k$ becomes $\frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \cdots$ which clearly diverges. So, the only hope for convergence is that the series $\sum a_k$ "eventually" becomes $0 + 0 + 0 + \cdots \implies \lim_{k\to\infty} a_k = 0$
- WARNING: Just because $\lim_{k\to\infty} a_k = 0$ does not necessarily mean that the series $\sum a_k$ converges.

INTEGRAL TEST: Suppose $a_k = f(k)$ for $k = N, N+1, N+2, \ldots$ s.t. f is **continuous** & **positive**.
Then: $\int_N^{\infty} f(x)\,dx$ converges (diverges) $\implies$ positive series $\sum_{k=N}^{\infty} a_k$ converges (diverges).

- NOTE: $\int_N^{\infty} f(x)\,dx < \infty \implies \int_N^{\infty} f(x)\,dx$ converges. $\quad \int_N^{\infty} f(x)\,dx = \infty$ or DNE $\implies \int_N^{\infty} f(x)\,dx$ diverges.
- REMARK: Series involving **factorials** (e.g. $k!$) are disqualified since the **Gamma Function** $\Gamma(\alpha)$ is too complicated.
- INTEGRAL DOMINANCE RULE:
  - ⋆ (IDR) $\quad f, g \in C[N,\infty)$ s.t. $f(x) \leq g(x) \;\; \forall x \in [N,\infty) \implies \int_N^{\infty} f(x)\,dx \leq \int_N^{\infty} g(x)\,dx$
  - ⋆ Sometimes the initial integral is hard to evaluate, so using the Dominance Rule often leads to simpler integrals.
  - ⋆ See the 8.3 Slides, 8.4 Slides, or the 8.4 Outline for a list of useful inequalities.

p-SERIES TEST: $p > 1 \implies p$-series $\sum_{k=1}^{\infty} \frac{1}{k^p}$ converges. $\quad p \leq 1 \implies p$-series $\sum_{k=1}^{\infty} \frac{1}{k^p}$ diverges.

EX 8.3.1: Test the series $\sum_{k=1}^{\infty} k \sin\left(\frac{1}{k}\right)$ for convergence.

EX 8.3.2: Test the series $\sum_{k=1}^{\infty} \frac{1}{e^k + e^{-k}}$ for convergence.

EX 8.3.3: Test the series $\sum_{k=2}^{\infty} \frac{\ln k}{\sqrt[3]{k}}$ for convergence.

EX 8.3.4: Test the series $\sum_{k=1}^{\infty} \frac{20k^2}{\sqrt{k^5}}$ for convergence.

EX 8.3.5: Test the series $\sum_{k=3}^{\infty} \frac{1}{5k^2\left(\sqrt[4]{k^3}\right)}$ for convergence.

EX 8.3.6: Test the series $\sum_{k=1}^{\infty} \left[\frac{1}{k} - \frac{1}{3^k}\right]$ for convergence.