POSITIVE SERIES: COMPARISON TESTS [SST 8.4]

DIRECT COMPARISON TEST:

$0 \le a_k \le c_k$ AND $\sum c_k$ converges $\implies \sum a_k$ converges
$b_k \ge d_k \ge 0$ AND $\sum d_k$ diverges $\implies \sum b_k$ diverges
where c_k, d_k are **simpler in form than** a_k, b_k

- KEY IDEA: Bound a complicated series with a simpler series whose convergence is known.
- DISADVANTAGE: Easy to pick the "wrong" series to compare with, causing the test to fail.

USEFUL INEQUALITIES FOR DIRECT COMPARISON TEST: $\left(k \in \mathbb{N}\right)$

- $-1 \le \cos x \le 1$ $\quad -1 \le \sin x \le 1$ $\quad -\frac{\pi}{2} < \arctan x < \frac{\pi}{2}$
- **For** $x \in \mathbb{R}$: $x^2 \ge 0,\ x^4 \ge 0,\ \cdots,\ x^{2n} \ge 0,\ 2^x > 0,\ e^x > 0,\ x^x \ge 0$
- **For** $x \ge 0$: $\sqrt{x} \ge 0,\ \sqrt[3]{x} \ge 0,\ \sqrt[4]{x} \ge 0,\ \cdots,\ \sqrt[n]{x} \ge 0,\ x^p \ge 0$
- **For** $x \ge 1$: $\sqrt{x} \ge 1,\ \sqrt[3]{x} \ge 1,\ \sqrt[4]{x} \ge 1,\ \cdots,\ \sqrt[n]{x} \ge 1,\ x^p \ge 1,\ \log_2 x \ge 0,\ \ln x \ge 0,\ \log x \ge 0$
- $A < B \implies -A > -B$ $\quad A > B \implies -A < -B$ $\quad A \le B \implies -A \ge -B$ $\quad A \ge B \implies -A \le -B$
- $A, M, m > 0$ s.t. $M > m \implies AM > Am$ and $\frac{A}{M} < \frac{A}{m}$
- $A, x > 0 \implies A + x > A \implies \frac{1}{A+x} < \frac{1}{A}$
- $A > x > 0 \implies A - x < A \implies \frac{1}{A-x} > \frac{1}{A}$
- f is **positive & increasing** on $[A, B]$ AND $0 < A < B \implies 0 < f(A) < f(B)$
- f is **positive & decreasing** on $[A, B]$ AND $0 < A < B \implies f(A) > f(B) > 0$
- **Eventually** (i.e. as $x, k \to \infty$):
  $\cdots \le \ln(\ln x) \le \log_{100} x \le \ln x \le \log_2 x \le \sqrt[100]{x} \le \sqrt[3]{x} \le \sqrt{x} \le x \le x^2 \le x^{100} \le 2^x \le e^x \le 100^x \le k! \le x^x \le x^{x^x} \le \cdots$

LIMIT COMPARISON TEST:

$\lim_{k\to\infty} \frac{a_k}{c_k} = 0$ AND $\sum c_k$ converges $\implies \sum a_k$ converges

$\lim_{k\to\infty} \frac{a_k}{b_k} = L \in (0, \infty) \implies \sum a_k$ and $\sum b_k$ either both converge or both diverge

$\lim_{k\to\infty} \frac{a_k}{d_k} = \infty$ AND $\sum d_k$ diverges $\implies \sum a_k$ diverges
where b_k, c_k, d_k are **simpler in form than** a_k

- KEY IDEA: Compare a complicated series with a simpler series that "looks like" the complicated one.
- ADVANTAGE: Uses limits instead of inequalities & often works when Direct CT fails.
- DISADVANTAGE: Some series are easy with Direct CT but hard or impossible with Limit CT. (EX 8.4.1 & 8.4.2)

SIMPLE SERIES TO COMPARE WITH:

Simple Convergent Series:	Simple Divergent Series:
Geometric Series $\sum_{k=0}^{\infty} r^k$ with $\lvert r\rvert < 1$	Geometric Series $\sum_{k=0}^{\infty} r^k$ with $\lvert r\rvert \ge 1$
p-series $\sum_{k=1}^{\infty} \frac{1}{k^p}$ with $p > 1$	p-series $\sum_{k=1}^{\infty} \frac{1}{k^p}$ with $p \le 1$

ADVICE:

- One can argue that this is the hardest section involving Series Tests.
- The key to picking the right series to compare with is **experience**.
  - Therefore, understand all the examples here, in the book, and in the HW.
  - It's also advised to attempt some of the problems in the book.

EX 8.4.1: Test the series $\sum_{k=1}^{\infty} \frac{3+\cos k}{k^2}$ for convergence.

EX 8.4.2: Test the series $\sum_{k=1}^{\infty} \frac{9k^2-1}{3k^5+k^2+4}$ for convergence.

EX 8.4.3: Test the series $\sum_{k=6}^{\infty} \frac{\ln k}{k-5}$ for convergence.

EX 8.4.4: Test the series $\sum_{k=1}^{\infty} \frac{2}{3k+\sqrt{k}}$ for convergence.

EX 8.4.5: Test the series $\sum_{k=1}^{\infty} \frac{\ln k}{k+5}$ for convergence.

EX 8.4.6: Test the series $\sum_{k=2}^{\infty} \frac{1}{(\ln k)^2}$ for convergence.

EX 8.4.7: Test the series $\sum_{k=1}^{\infty} \frac{k+3}{k(k+8)}$ for convergence.

EX 8.4.8: Test the series $\sum_{k=2}^{\infty} \frac{1}{1+\ln k}$ for convergence.

EX 8.4.9: Test the series $\sum_{k=1}^{\infty} \frac{k}{(k+1)2^{k-1}}$ for convergence.

EX 8.4.10: Test the series $\sum_{k=2}^{\infty} \frac{1}{k^3\left[\ln(k+7)-2\right]}$ for convergence.