POSITIVE SERIES: RATIO TEST, ROOT TEST [SST 8.5]

RATIO TEST:

$$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} < 1 \implies \text{positive series } \sum a_k \text{ converges.}$$
$$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} > 1 \implies \text{positive series } \sum a_k \text{ diverges.}$$
$$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \infty \implies \text{positive series } \sum a_k \text{ diverges.}$$
$$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = 1 \text{ or DNE} \implies \text{Ratio Test fails.}$$

- KEY IDEA: The **ratio** $\dfrac{a_{k+1}}{a_k}$ measures one sense of "how fast the terms a_k **decrease to zero**."
- ADVANTAGES:
  - Involves taking a limit and simple division, so easy to use.
  - Works nicely with:
    - **factorials** (e.g. $k!$)
    - **simple powers** (e.g. k^3)
    - **simple exponentials** (e.g. 2^k)
    - **product chains** (e.g. $1 \cdot 3 \cdot 5 \cdots (2k-1)$).
- DISADVANTAGES:
  - Fails for many series.
  - Useless with many **trig expressions** since: $\lim_{k\to\infty} \sin k = \text{DNE}$ $\quad \lim_{k\to\infty} \cos k = \text{DNE}$
  - Tedious to use with most **rational functions of polynomials**. $\left(\text{e.g. } a_k = \frac{k^3-2k^2+k+1}{k^4+k^3+k^2+2}\right)$

ROOT TEST:

$$\lim_{k\to\infty} \sqrt[k]{a_k} < 1 \implies \text{positive series } \sum a_k \text{ converges.}$$
$$\lim_{k\to\infty} \sqrt[k]{a_k} > 1 \implies \text{positive series } \sum a_k \text{ diverges.}$$
$$\lim_{k\to\infty} \sqrt[k]{a_k} = \infty \implies \text{positive series } \sum a_k \text{ diverges.}$$
$$\lim_{k\to\infty} \sqrt[k]{a_k} = 1 \text{ or DNE} \implies \text{Root Test fails.}$$

- KEY IDEA: The **root** $\sqrt[k]{a_k}$ measures another sense of "how fast the terms a_k **decrease to zero**."
- ADVANTAGES:
  - Works nicely with:
    - **heavy powers of** k $\left(\text{e.g. } [f(k)]^k, \text{ where } f \text{ is a } \textbf{continuous function}\right)$
    - **heavy exponentials** $\left(\text{e.g. } 9^{k^2}, 7^{\ln k}, 3^{\sin k}, \dots\right)$.
- DISADVANTAGES:
  - Fails for many series.
  - Useless with **factorials** (e.g. $k!$).
- USEFUL LIMITS: $\lim_{k\to\infty} \sqrt[k]{2} = 1$ $\quad \lim_{k\to\infty} \sqrt[k]{k} = 1$

FACTORIALS: $\left(k \in \bar{\mathbb{Z}}_+ := \{0,1,2,3,4,5,\cdots\}\right)$

- $k! := k(k-1)(k-2)\cdots(3)(2)(1)$ $\quad 0! := 1$
- CAUTION: $(k+3)! \neq k!+3$ or $k!+3!$, rather $(k+3)! = (k+3)(k+2)(k+1)k!$
- CAUTION: $(3k)! \neq 3k!$ or $3!k!$, rather $(3k)! = (3k)(3k-1)(3k-2)(3k-3)\cdots(k+2)(k+1)k!$
- CAUTION: $k!k! \neq \left(k^2\right)!$, rather $k!k! = (k!)^2$

EX 8.5.1: Test the series $\sum_{k=1}^{\infty} \frac{k^2 2^{k+1}}{3^k}$ for convergence.

EX 8.5.2: Test the series $\sum_{k=1}^{\infty} \frac{6^k k! k!}{(2k)!}$ for convergence.

EX 8.5.3: Test the series $\sum_{k=1}^{\infty} \frac{k!}{1 \cdot 3 \cdot 5 \cdots (2k-1)}$ for convergence.

EX 8.5.4: Test the series $\sum_{k=1}^{\infty} \frac{e^{5k}}{k^k}$ for convergence.

EX 8.5.5: Test the series $\sum_{k=1}^{\infty} \frac{k^k}{5^{k^2}}$ for convergence.

EX 8.5.6: Test the series $\sum_{k=1}^{\infty} \left[\frac{k^2 e^k}{\ln\left(\pi + \frac{1}{k}\right)} \right]^k$ for convergence.