ALTERNATING SERIES & ABSOLUTE CONVERGENCE [SST 8.6]

ALTERNATING SERIES:

- An **alternating series** has one of the forms: $\sum(-1)^k a_k$, $\sum(-1)^{k+1}a_k$ or $\sum(-1)^{k-1}a_k$, where $a_k \geq 0 \quad \forall k$

Series	Expansion	**Sign Pattern**
$\sum_{k=0}^{\infty} a_k$	$a_0 + a_1 + a_2 + a_3 + a_4 + a_5 + \cdots$	$+++++++++\cdots$
$\sum_{k=0}^{\infty}(-1)^{k-1}a_k$	$-a_0 + a_1 - a_2 + a_3 - a_4 + a_5 - \cdots$	$-+-+-+-+\cdots$
$\sum_{k=0}^{\infty}(-1)^{k}a_k$	$a_0 - a_1 + a_2 - a_3 + a_4 - a_5 + \cdots$	$+-+-+-+-\cdots$
$\sum_{k=0}^{\infty}(-1)^{k+1}a_k$	$-a_0 + a_1 - a_2 + a_3 - a_4 + a_5 - \cdots$	$-+-+-+-+\cdots$
$\sum_{k=0}^{\infty}(-1)^{k(k+1)/2}a_k$	$a_0 - a_1 - a_2 + a_3 + a_4 - a_5 - \cdots$	$+--++--+\cdots$

- Note that some powers of -1 can be simplified: $(-1)^{3k+3} = (-1)^{3(k+1)} = \left[(-1)^3\right]^{k+1} = (-1)^{k+1}$
- But not all powers of -1 indicate that a series is alternating:
  - e.g. $(-1)^{2k+2} = (-1)^{2(k+1)} = \left[(-1)^2\right]^{k+1} = 1^{k+1} = 1 \implies \sum(-1)^{2k+2}a_k$ is a **positive series** (not alternating)
  - e.g. $\sum 2^{-k-(-1)^k}$ is a **positive series** even though it has a $(-1)^k$ term.
- Other powers of -1 indicate that a series is **neither positive nor alternating**: $(-1)^{k(k+1)/2}$ (see above & EX 8.6.4)

ALTERNATING SERIES TEST: Let $a_k > 0 \quad \forall k$ s.t. $\lim\limits_{k\to\infty} a_k = 0$ AND $\{a_k\}$ is eventually decreasing.
Then alternating series $\sum(-1)^k a_k$, $\sum(-1)^{k+1}a_k$ and $\sum(-1)^{k-1}a_k$ all converge.

ABSOLUTE CONVERGENCE:

- Now, suppose $\sum a_k$ is not necessarily positive nor alternating. (see EX 8.6.4 & EX 8.6.5)
- Series $\sum a_k$ is said to **converge absolutely** if the corresponding positive series $\sum |a_k|$ converges.
- Series $\sum a_k$ is said to **converge conditionally** if $\sum a_k$ converges, but $\sum |a_k|$ diverges.
- Especially if $\sum a_k$ is **neither positive nor alternating**, the only applicable test is the **Absolute Convergence Test**.
- If $\sum a_k$ is **neither positive nor alternating** and the Absolute Convergence Test **fails**, then $\sum a_k$ is essentially a **graduate-level series**, and will not be considered in Calculus II as very advanced methods (Dirichlet's Test, Abel's Test, ...) seen in Advanced Calculus and graduate courses are needed. e.g. $\sum\limits_{k=1}^{\infty} \frac{\sin k}{k}$

ABSOLUTE CONVERGENCE TEST: Positive series $\sum |a_k|$ converges $\implies$ series $\sum a_k$ converges.

ABSOLUTE CONVERGENCE IMPLIES SERIES CAN BE REARRANGED:

- RECALL: Addition is **commutative**: $3+5=5+3$ and **associative**: $3+(5+7)=(3+5)+7$
- So, for **finite series**, the terms can be **rearranged**: e.g. $\sum\limits_{k=1}^{3} k = 1+2+3 = 1+3+2 = 2+3+1 = 3+2+1 = 6$
- An **absolutely convergent series** also can be **rearranged**:
  e.g. $\sum\limits_{k=1}^{\infty}\frac{(-1)^k}{k^2} = -1 + \frac{1}{4} - \frac{1}{9} + \frac{1}{16} - \frac{1}{25} + \frac{1}{36} - \frac{1}{49} + \cdots = \left(-1 - \frac{1}{9}\right) + \left(\frac{1}{4} + \frac{1}{16}\right) + \left(-\frac{1}{25} - \frac{1}{49}\right) + \cdots$
- **Rearranging** a **conditionally convergent series** yields **different sums**:
  e.g. $\sum\limits_{k=1}^{\infty}\frac{(-1)^{k+1}}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots = \ln 2$, but $\left(1 - \frac{1}{2}\right) - \frac{1}{4} + \left(\frac{1}{3} - \frac{1}{6}\right) - \frac{1}{8} + \cdots = \frac{1}{2}\ln 2$
- **Rearranging** a **divergent series** yields **downright weird results**:
  e.g. $\sum\limits_{k=1}^{\infty}(-1)^k = -1+1-1+1-1+\cdots$ diverges by oscillation, but $(-1+1)+(-1+1)+(-1+1)+\cdots = 0$
  and $(1+1-1)+(1+1-1)+(1+1-1)+\cdots = +\infty$ and $(1-1-1)+(1-1-1)+(1-1-1)+\cdots = -\infty$

EX 8.6.1: Test the series $\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$ for convergence and absolute convergence.

EX 8.6.2: Test the series $\sum_{k=2}^{\infty} (-1)^{k+1} \left(\frac{6k^2+4}{7k^2-1} \right)^{1/3}$ for convergence and absolute convergence.

EX 8.6.3: Test the series $\sum_{k=3}^{\infty} \frac{(-1)^k e^{-3k}}{k!\sqrt{k}}$ for convergence and absolute convergence.

EX 8.6.4: Test the series $\sum_{k=1}^{\infty} \frac{(-1)^{k(k+1)/2}}{k^4}$ for convergence and absolute convergence.

EX 8.6.5: Test the series $\sum_{k=1}^{\infty} \frac{\cos(k\pi/3)}{k!}$ for convergence and absolute convergence.