POWER SERIES [SST 8.7]

DEFINITION:

- A **power series** has the form: $\sum_{k=0}^{\infty} a_k(x-c)^k = a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \cdots$
- A power series with $c=0$ simplifies to: $\sum_{k=0}^{\infty} a_k x^k = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$
- A **polynomial** is a **finite power series** : e.g. $x^3 - 2x^2 + 3x - 4 = \sum_{k=0}^{\infty} a_k x^k$ with $\begin{cases} a_0 = -4, a_1 = 3, a_2 = -2, a_3 = 1 \\ a_4 = a_5 = a_6 = a_7 = \cdots = 0 \end{cases}$

CONVERGENCE OF A POWER SERIES:

- Given a **power series** $\sum_{k=0}^{\infty} a_k(x-c)^k$, exactly one of the following is true:

$\sum a_k(x-c)^k$ converges for all x	$\iff$	**set of convergence** is $\mathbb{R}$	$\iff$	**radius of conv.** is ∞
$\sum a_k(x-c)^k$ converges only for $x=c$	$\iff$	**set of convergence** is $\{c\}$	$\iff$	**radius of conv.** is 0
$\sum a_k(x-c)^k$converges $\forall x \in (c-R, c+R)$	$\iff$	**set of convergence** is $(c-R, c+R)$	$\iff$	**radius of conv.** is R
$\sum a_k(x-c)^k$converges $\forall x \in [c-R, c+R)$	$\iff$	**set of convergence** is $[c-R, c+R)$	$\iff$	**radius of conv.** is R
$\sum a_k(x-c)^k$converges $\forall x \in (c-R, c+R]$	$\iff$	**set of convergence** is $(c-R, c+R]$	$\iff$	**radius of conv.** is R
$\sum a_k(x-c)^k$converges $\forall x \in [c-R, c+R]$	$\iff$	**set of convergence** is $[c-R, c+R]$	$\iff$	**radius of conv.** is R

  NOTE: For the bottom four cases above, the power series **converges absolutely** on the **open interval** $(c-R, c+R)$.

- To find the **radius of convergence**, use the **Ratio Test** or **Root Test** on $\sum \left|a_k(x-c)^k\right|$:
  - Ratio Test: solve the **inequality** $\lim_{k\to\infty} \left|\frac{a_{k+1}(x-c)^{k+1}}{a_k(x-c)^k}\right| < 1$ for x
  - Root Test: solve the **inequality** $\lim_{k\to\infty} \sqrt[k]{|a_k(x-c)^k|} < 1$ for x
  - If, upon simplifying, you get a **true statement** like $0 < 1$, then the **radius of convergence** $R = \infty$
  - If, upon simplifying, you get a **false statement** like $3 < 1$, then the **radius of convergence** $R = 0$
- To determine convergence on the **boundary** of interval $(c-R, c+R)$, that is, at $x = c-R$ and $x = c+R$, **test the series for convergence at each endpoint**.

PROPERTIES OF A POWER SERIES:

- A **power series** $\sum_{k=0}^{\infty} a_k(x-c)^k$ with **radius of convergence** $R > 0$:
  - Is **infinitely differentiable** on its **interval of absolute convergence** $\iff$ $\sum a_k(x-c)^k \in C^{\infty}(c-R, c+R)$
  - Can be **differentiated term by term** on its **interval of absolute convergence** $(c-R, c+R)$:
    $f'(x) = \frac{d}{dx}\left[\sum_{k=0}^{\infty} a_k(x-c)^k\right] = \sum_{k=0}^{\infty} \frac{d}{dx}\left[a_k(x-c)^k\right] = \sum_{k=1}^{\infty} k a_k (x-c)^{k-1}$
  - Can be **integrated term by term** on its **interval of absolute convergence** $(c-R, c+R)$:
    $\int f(x)\ dx = \int \left(\sum_{k=0}^{\infty} a_k(x-c)^k\right)\ dx = \sum_{k=0}^{\infty}\left(\int a_k(x-c)^k\ dx\right) = \sum_{k=0}^{\infty} \frac{a_k}{k+1}(x-c)^{k+1} + C$
    $\int_a^b f(x)\ dx = \int_a^b \left(\sum_{k=0}^{\infty} a_k(x-c)^k\right)\ dx = \left(\sum_{k=0}^{\infty} \int_a^b a_k(x-c)^k\ dx\right) = \sum_{k=0}^{\infty}\left[\frac{a_k}{k+1}(x-c)^{k+1}\right]_{x=a}^{x=b}$
  - Can have its terms **rearranged without changing its sum** [SST 8.6]
- In a nutshell, a **power series behaves like a polynomial on its interval of absolute convergence**.
- Power series can represent **elementary functions**.
  - This means we can now **integrate nonelementary integrals** such as $\int e^{x^2}\ dx$ [SST 8.8]

EX 8.7.1: On what set does the series $\sum_{k=1}^{\infty} \frac{x^k}{k^k}$ converge? What is the radius of convergence?

EX 8.7.2: On what set does the series $\sum_{k=0}^{\infty} \sqrt[3]{k}(3k)!x^k$ converge? What is the radius of convergence?

EX 8.7.3: On what set does the series $\sum_{k=1}^{\infty} \frac{(3x)^{3k}}{\sqrt{k}}$ converge? What is the radius of convergence?

EX 8.7.4: On what set does the series $\sum_{k=1}^{\infty} \frac{(\ln k)(x-2)^k}{k}$ converge? What is the radius of convergence?

EX 8.7.5: Define f by the power series: $f(x) := \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \frac{1}{6!}x^6 + \cdots$

(a) Find the **set of convergence** & **radius of convergence** of f.

(b) Compute $f'(x)$.

(c) Compute $\int_0^x f(t)\, dt$.

(d) Based on your answer in part (b), which **elementary function** must f be?

(e) Compute the **sum** of the series $2 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \frac{1}{720} + \cdots$
(Notice that this series is neither geometric nor telescoping....)

EX 8.7.6: Define g by the power series: $g(x) := \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \cdots$

(a) Find the **set of convergence** & **radius of convergence** of g.

(b) Compute $g'(x)$.

(c) Compute $g''(x)$.

(d) Based on your answer in part (c), which **elementary function** must g be?

(e) Compute the **sum** of the series $\frac{\pi}{4} - \frac{\pi^3}{3!4^3} + \frac{\pi^5}{5!4^5} - \frac{\pi^7}{7!4^7} + \frac{\pi^9}{9!4^9} - \frac{\pi^{11}}{11!4^{11}} + \cdots$
(Notice that this series is neither geometric nor telescoping....)

EX 8.7.7:

(a) Using the fact that $\arctan x = \int_0^x \frac{dt}{1+t^2}$, find a **power series representation** for $\arctan x$.

(b) Find the **set of convergence** & **radius of convergence** of the power series for $\arctan x$.

(c) Compute the **sum** of the series $\frac{9}{5}\sqrt{3} - \frac{27}{7}\sqrt{3} + 9\sqrt{3} - \frac{243}{11}\sqrt{3} + \cdots$

(Notice that this series is neither geometric nor telescoping....)

(d) Find a **series representation** for π.