POWER SERIES [SST 8.7]

DEFINITION:

- A power series has the form: $\sum_{k=0}^{\infty} a_k (x-c)^k = a_0 + a_1 (x-c) + a_2 (x-c)^2 + a_3 (x-c)^3 + \cdots$
- A power series with c = 0 simplifies to: $\sum_{k=0}^{\infty} a_k x^k = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$
- A polynomial is a finite power series : e.g. $x^3 2x^2 + 3x 4 = \sum_{k=0}^{\infty} a_k x^k$ with $\begin{cases} a_0 = -4, a_1 = 3, a_2 = -2, a_3 = 1 \\ a_4 = a_5 = a_6 = a_7 = \dots = 0 \end{cases}$

CONVERGENCE OF A POWER SERIES:

• Given a power series $\sum_{k=0}^{\infty} a_k(x-c)^k$, exactly one of the following is true: $\sum a_k(x-c)^k$ converges for all $x \quad \iff \quad$ set of convergence is $\mathbb{R} \quad \iff \quad$ radius of conv. is ∞ $\sum a_k(x-c)^k$ converges only for $x=c \quad \iff \quad$ set of convergence is $\{c\} \quad \iff \quad$ radius of conv. is 0 $\sum a_k(x-c)^k$ converges $\forall x \in (c-R, c+R) \quad \iff \quad$ set of convergence is $(c-R, c+R) \quad \iff \quad$ radius of conv. is R $\sum a_k(x-c)^k$ converges $\forall x \in [c-R, c+R) \quad \iff \quad$ set of convergence is $[c-R, c+R) \quad \iff \quad$ radius of conv. is R $\sum a_k(x-c)^k$ converges $\forall x \in [c-R, c+R] \quad \iff \quad$ set of convergence is $(c-R, c+R) \quad \iff \quad$ radius of conv. is R $\sum a_k(x-c)^k$ converges $\forall x \in [c-R, c+R] \quad \iff \quad$ set of convergence is $(c-R, c+R] \quad \iff \quad$ radius of conv. is R $\sum a_k(x-c)^k$ converges $\forall x \in [c-R, c+R] \quad \iff \quad$ set of convergence is $[c-R, c+R] \quad \iff \quad$ radius of conv. is R $\sum a_k(x-c)^k$ converges $\forall x \in [c-R, c+R] \quad \iff \quad$ set of convergence is $[c-R, c+R] \quad \iff \quad$ radius of conv. is R $\sum a_k(x-c)^k$ converges $\forall x \in [c-R, c+R] \quad \iff \quad$ set of convergence is $[c-R, c+R] \quad \iff \quad$ radius of conv. is R $\sum a_k(x-c)^k$ converges $\forall x \in [c-R, c+R] \quad \iff \quad$ set of convergence is $[c-R, c+R] \quad \iff \quad$ radius of conv. is R $\sum a_k(x-c)^k$ converges $\forall x \in [c-R, c+R] \quad \iff \quad$ set of convergence is $[c-R, c+R] \quad \iff \quad$ radius of conv. is R $\sum a_k(x-c)^k$ converges $\forall x \in [c-R, c+R] \quad \iff \quad$ set of convergence is $[c-R, c+R] \quad \iff \quad$ radius of conv. is R $\sum a_k(x-c)^k$ converges $\forall x \in [c-R, c+R] \quad \iff \quad$ set of convergence is $[c-R, c+R] \quad \iff \quad$ radius of conv. is R $\sum a_k(x-c)^k$ converges $\forall x \in [c-R, c+R] \quad \iff \quad$ set of convergence is $[c-R, c+R] \quad \iff \quad$ radius of conv. is R $\sum a_k(x-c)^k$ converges $\forall x \in [c-R, c+R] \quad \iff \quad$ set of convergence is $[c-R, c+R] \quad \iff \quad$ radius of conv. is R $\sum a_k(x-c)^k$ converges $\forall x \in [c-R, c+R] \quad \iff \quad$ radius of conv. is R $\sum a_k(x-c)^k$ converges $\forall x \in [c-R, c+R] \quad \iff \quad$ radius of conv. is R $\sum a_k(x-c)^k$ convergence is $[c-R, c+R] \quad \iff \quad$ radius of conv. is R $\sum a_k(x-c)^k$ convergence is [c-R

- To find the radius of convergence, use the Ratio Test or Root Test on $\sum |a_k(x-c)^k|$:
 - Ratio Test: solve the **inequality** $\lim_{k \to \infty} \left| \frac{a_{k+1}(x-c)^{k+1}}{a_k(x-c)^k} \right| < 1 \text{ for } x$
 - Root Test: solve the **inequality** $\lim_{k\to\infty} \sqrt[k]{|a_k(x-c)^k|} < 1$ for x
 - If, upon simplifying, you get a true statement like 0 < 1, then the radius of convergence $R = \infty$
 - If, upon simplifying, you get a false statement like 3 < 1, then the radius of convergence R = 0
- To determine convergence on the **boundary** of interval (c R, c + R), that is, at x = c R and x = c + R, test the series for convergence at each endpoint.

PROPERTIES OF A POWER SERIES:

- A power series $\sum_{k=0}^{\infty} a_k (x-c)^k$ with radius of convergence R > 0:
 - Is infinitely differentiable on its interval of absolute convergence $\iff \sum a_k(x-c)^k \in C^{\infty}(c-R,c+R)$
 - Can be differentiated term by term on its interval of absolute convergence (c R, c + R):

$$f'(x) = \frac{d}{dx} \left[\sum_{k=0}^{\infty} a_k (x-c)^k \right] = \sum_{k=0}^{\infty} \frac{d}{dx} \left[a_k (x-c)^k \right] = \sum_{k=1}^{\infty} k a_k (x-c)^{k-1}$$

- Can be integrated term by term on its interval of absolute convergence (c - R, c + R):

$$\int f(x) \, dx = \int \left(\sum_{k=0}^{\infty} a_k (x-c)^k\right) \, dx = \sum_{k=0}^{\infty} \left(\int a_k (x-c)^k \, dx\right) = \sum_{k=0}^{\infty} \frac{a_k}{k+1} (x-c)^{k+1} + C$$

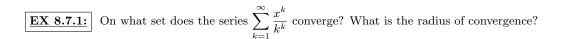
$$\int_a^b f(x) \, dx = \int_a^b \left(\sum_{k=0}^{\infty} a_k (x-c)^k\right) \, dx = \left(\sum_{k=0}^{\infty} \int_a^b a_k (x-c)^k \, dx\right) = \sum_{k=0}^{\infty} \left[\frac{a_k}{k+1} (x-c)^{k+1}\right]_{x=a}^{x=b}$$

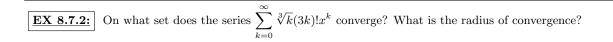
$$\int_a^b f(x) \, dx = \int_a^b \left(\sum_{k=0}^{\infty} a_k (x-c)^k\right) \, dx = \left(\sum_{k=0}^{\infty} \int_a^b a_k (x-c)^k \, dx\right) = \sum_{k=0}^{\infty} \left[\frac{a_k}{k+1} (x-c)^{k+1}\right]_{x=a}^{x=b}$$

- Can have its terms rearranged without changing its sum [SST 8.6]
- In a nutshell, a power series behaves like a polynomial on its interval of absolute convergence.
- Power series can represent **elementary functions**.

- This means we can now integrate nonelementary integrals such as $\int e^{x^2} dx$ [SST 8.8]

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<u>EX 8.7.3</u> On what set does the series \sum_{k=1}^{\infty} \frac{(3x)^{3k}}{\sqrt{k}} converge? What is the radius of convergence?
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<u>EX 8.7.4</u>: On what set does the series $\sum_{k=1}^{\infty} \frac{(\ln k)(x-2)^k}{k}$ converge? What is the radius of convergence?

EX 8.7.5: Define f by the power series: $f(x) := \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \frac{1}{6!}x^6 + \cdots$

(a) Find the set of convergence & radius of convergence of f.

(b) Compute f'(x).

(c) Compute $\int_0^x f(t) dt$.

(d) Based on your answer in part (b), which elementary function must f be?

(e) Compute the **sum** of the series $2 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \frac{1}{720} + \cdots$ (Notice that this series is neither geometric nor telescoping....)

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EX 8.7.6 Define g by the power series:
$$g(x) := \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \cdots$$

(a) Find the set of convergence & radius of convergence of g.

(b) Compute g'(x).

(c) Compute g''(x).

(d) Based on your answer in part (c), which elementary function must g be?

(e) Compute the **sum** of the series $\frac{\pi}{4} - \frac{\pi^3}{3!4^3} + \frac{\pi^5}{5!4^5} - \frac{\pi^7}{7!4^7} + \frac{\pi^9}{9!4^9} - \frac{\pi^{11}}{11!4^{11}} + \cdots$ (Notice that this series is neither geometric nor telescoping....)

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EX 8.7.7:

(a) Using the fact that $\arctan x = \int_0^x \frac{dt}{1+t^2}$, find a **power series representation** for $\arctan x$.

(b) Find the set of convergence & radius of convergence of the power series for $\arctan x$.

(c) Compute the **sum** of the series $\frac{9}{5}\sqrt{3} - \frac{27}{7}\sqrt{3} + 9\sqrt{3} - \frac{243}{11}\sqrt{3} + \cdots$ (Notice that this series is neither geometric nor telescoping....)

(d)

Find a series representation for π .