TAYLOR SERIES [SST 8.8]

TAYLOR SERIES:

- Every function $f \in C^{\infty}(c-R, c+R)$ has a unique **Taylor series about** $x = c$ of the form:

  $$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k = f(c) + \frac{f'(c)}{1!}(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \frac{f^{(4)}(c)}{4!}(x-c)^4 + \cdots$$

  which is a special type of **power series** that **converges absolutely** to $f(x)$ for all $x \in (c-R, c+R)$ where R is the **radius of convergence**.

- NOTE: It's possible that $f(x)$ exists for x-values outside of the **interval of convergence** of its Taylor series.
  - e.g. The **Taylor series about** $x = 0$ for $\ln(1+x)$ **converges** for all $x \in (-1, 1]$, yet $\ln(1+2)$ is defined!!
  - i.e. A Taylor series for $f(x)$ is a **very poor approximation** to $f(x)$ for all x **outside the interval of convergence**.

- TERMINOLOGY: A **Maclaurin series** is just a **Taylor series about** $x = 0$:

  $$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \cdots$$

USEFUL TOOLS FOR FINDING A TAYLOR SERIES:

- Finding the derivatives $f(c), f'(c), f''(c), f'''(c), f^{(4)}(c), \ldots$ (always available, but not always fast)
- Clever **substitution** into a known Taylor series
- Clever manipulation of a **geometric series**
- **Differentiating** a known Taylor series
- **Integrating** a known Taylor series
- Clever use of **trig identities**
- **Multiplying** a Taylor series by a **monomial** (single-term polynomial)
- **Dividing** a Taylor series by a **monomial** (single-term polynomial)
- **Multiplying** two Taylor series
- **Dividing** two Taylor series (this is subtle, so not to be considered here)
- Clever manipulation of a **Binomial series** (see below)

BINOMIAL SERIES:

- $(1+x)^{\alpha} = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \cdots$ where $\alpha \in \mathbb{R}$ and $|x| < 1$.

APPLICATIONS OF TAYLOR SERIES:

- Finding **limits** that otherwise would **require far too many iterations of L'Hôpital's Rule**
- Evaluation of **nonelementary integrals**
- Solution to advanced **differential equations** (not considered here)
- Polynomial approximation of a complicated function and its error analysis (not considered here)
- Forms a central foundation of the **theory of functions of complex variables** (not considered here)

EX 8.8.1: Find the first three nonzero terms in the Taylor series about $x = 0$ for e^x.

EX 8.8.2: Find the first three nonzero terms in the Taylor series about $x = 0$ for $\sin x$.

EX 8.8.3: Find the first three nonzero terms in the Taylor series about $x = 1$ for $\ln x$.

EX 8.8.4: Find the first three nonzero terms in the Taylor series about $x = 0$ for e^{x^2}.

EX 8.8.5: Find the first three nonzero terms in the Taylor series about $x = 0$ for $x^3 e^x$.

EX 8.8.6: Find the first three nonzero terms in the Taylor series about $x = 0$ for $\cos x$.

EX 8.8.7: Find the first three nonzero terms in the Taylor series about $x = 0$ for $2 \sin x \cos x$.

EX 8.8.8: Find the first three nonzero terms in the Taylor series about $x = 0$ for $\dfrac{1}{1-x}$.

EX 8.8.9: Find the first three nonzero terms in the Taylor series about $x = 0$ for $\dfrac{1}{(1-x)^2}$.

EX 8.8.10: Find the first three nonzero terms in the Taylor series about $x = 0$ for $\dfrac{1}{1+x^2}$.

EX 8.8.11: Find the first three nonzero terms in the Taylor series about $x = 0$ for $\arctan x$.

EX 8.8.12: Find the first three nonzero terms in the Taylor series about $x = 0$ for $e^x \cos x$.

EX 8.8.13: Compute the limit $\lim\limits_{x \to 0} \dfrac{\sin x - x + \frac{1}{6}x^3 - \frac{1}{120}x^5}{x^7}$.

EX 8.8.14: Using Taylor series about $x = 0$, compute the first three terms of $\displaystyle\int e^{x^2}\, dx$.

EX 8.8.15: Using Taylor series about $x = 0$, compute the first three terms of $\displaystyle\int \frac{\sin x}{x}\, dx$.

EX 8.8.16: Using **Binomial series**, compute the first three terms of $\sqrt{1+x}$.

EX 8.8.17: Using **Binomial series**, compute the first three terms of $\sqrt[4]{3-2x}$.

EX 8.8.18: Using **Binomial series**, compute the first three terms of $\dfrac{x^5}{\sqrt[3]{1-x^2}}$.