VECTORS: DOT PRODUCT [SST 9.3]

DOT PRODUCT OF TWO VECTORS:

- In $\mathbb{R}^2$, $\mathbf{v} = \langle v_1, v_2 \rangle$ and $\mathbf{w} = \langle w_1, w_2 \rangle \implies$ **Dot product** $\mathbf{v} \cdot \mathbf{w} = v_1 w_1 + v_2 w_2$
- In $\mathbb{R}^3$, $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ and $\mathbf{w} = \langle w_1, w_2, w_3 \rangle \implies$ **Dot product** $\mathbf{v} \cdot \mathbf{w} = v_1 w_1 + v_2 w_2 + v_3 w_3$
- Notice that the dot product of two vectors is a scalar!

PROPERTIES OF DOT PRODUCTS:

$\Big($Let vectors $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \{\mathbb{R}^2, \mathbb{R}^3\}$ and scalar $c \in \mathbb{R}\Big)$

- $\mathbf{v} \cdot \mathbf{v} = ||\mathbf{v}||^2$
- $\vec{\mathbf{0}} \cdot \mathbf{v} = \mathbf{v} \cdot \vec{\mathbf{0}} = 0$
- $\mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v}$
- $c(\mathbf{v} \cdot \mathbf{w}) = (c\mathbf{v}) \cdot \mathbf{w} = \mathbf{v} \cdot (c\mathbf{w})$
- $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$

ANGLE BETWEEN VECTORS:

- $\mathbf{v} \cdot \mathbf{w} = ||\mathbf{v}||\,||\mathbf{w}|| \cos(\theta_{vw})$, where the **angle between the two vectors** $\theta_{vw} \in [0, \pi]$ and $\mathbf{v}, \mathbf{w} \in \{\mathbb{R}^2, \mathbb{R}^3\}$
- Nonzero vectors $\mathbf{v}, \mathbf{w}$ are **orthogonal** $\iff \mathbf{v} \cdot \mathbf{w} = 0$
- Nonzero vectors $\mathbf{v}, \mathbf{w}$ are **parallel** $\iff \mathbf{v} = t\mathbf{w}$ for some scalar $t \in \mathbb{R}$.

DIRECTION ANGLES & DIRECTION COSINES:

- In $\mathbb{R}^2$: $\Big($Here, vector $\mathbf{v} = \langle v_1, v_2 \rangle\Big)$
  - The **direction angle** θ of nonzero vector $\mathbf{v} \in \mathbb{R}^2$ is the smallest positive angle it makes with the positive x-axis.
  - The **direction cosine** of nonzero vector $\mathbf{v} \in \mathbb{R}^2$ is the cosine of the direction angle : $\cos\theta$
  - Hence, $\mathbf{v} \cdot \widehat{\mathbf{i}} = ||\mathbf{v}||\,||\widehat{\mathbf{i}}|| \cos\theta \implies \cos\theta = \dfrac{\mathbf{v} \cdot \widehat{\mathbf{i}}}{||\mathbf{v}||\,||\widehat{\mathbf{i}}||} = \dfrac{v_1}{||\mathbf{v}||}$, where **basis vector** $\widehat{\mathbf{i}} = \langle 1, 0 \rangle$
- In $\mathbb{R}^3$: $\Big($Here, vector $\mathbf{v} = \langle v_1, v_2, v_3 \rangle\Big)$
  - The **direction angle** α of nonzero vector $\mathbf{v} \in \mathbb{R}^3$ is the smallest positive angle it makes with the positive x-axis.
  - The **direction angle** β of nonzero vector $\mathbf{v} \in \mathbb{R}^3$ is the smallest positive angle it makes with the positive y-axis.
  - The **direction angle** γ of nonzero vector $\mathbf{v} \in \mathbb{R}^3$ is the smallest positive angle it makes with the positive z-axis.
  - The **direction cosines** of nonzero vector $\mathbf{v} \in \mathbb{R}^3$ are: $\cos\alpha, \cos\beta, \cos\gamma$
  - Hence, $\mathbf{v} \cdot \widehat{\mathbf{i}} = ||\mathbf{v}||\,||\widehat{\mathbf{i}}|| \cos\alpha \implies \cos\alpha = \dfrac{\mathbf{v} \cdot \widehat{\mathbf{i}}}{||\mathbf{v}||\,||\widehat{\mathbf{i}}||} = \dfrac{v_1}{||\mathbf{v}||}$, where **basis vector** $\widehat{\mathbf{i}} = \langle 1, 0, 0 \rangle$
  - Hence, $\mathbf{v} \cdot \widehat{\mathbf{j}} = ||\mathbf{v}||\,||\widehat{\mathbf{j}}|| \cos\beta \implies \cos\beta = \dfrac{\mathbf{v} \cdot \widehat{\mathbf{j}}}{||\mathbf{v}||\,||\widehat{\mathbf{j}}||} = \dfrac{v_2}{||\mathbf{v}||}$, where **basis vector** $\widehat{\mathbf{j}} = \langle 0, 1, 0 \rangle$
  - Hence, $\mathbf{v} \cdot \widehat{\mathbf{k}} = ||\mathbf{v}||\,||\widehat{\mathbf{k}}|| \cos\gamma \implies \cos\gamma = \dfrac{\mathbf{v} \cdot \widehat{\mathbf{k}}}{||\mathbf{v}||\,||\widehat{\mathbf{k}}||} = \dfrac{v_3}{||\mathbf{v}||}$, where **basis vector** $\widehat{\mathbf{k}} = \langle 0, 0, 1 \rangle$

PROJECTIONS:

- **Vector projection of v** onto $\mathbf{w}$: $\text{proj}_{\mathbf{w}}\mathbf{v} = \left(\dfrac{\mathbf{v} \cdot \mathbf{w}}{\mathbf{w} \cdot \mathbf{w}}\right)\mathbf{w}$
- **Scalar projection of v** onto $\mathbf{w}$: $\text{comp}_{\mathbf{w}}\mathbf{v} = ||\text{proj}_{\mathbf{w}}\mathbf{v}|| = \dfrac{\mathbf{v} \cdot \mathbf{w}}{||\mathbf{w}||}$

WORK AS A DOT PRODUCT:

- The **work** W done by a **constant force F** on an object moving along the line from point P to point Q is: $W = \mathbf{F} \cdot \mathbf{PQ}$

EX 9.3.1: (a) Find the dot product $\mathbf{v} \cdot \mathbf{w}$ if $\mathbf{v} = \langle 1, 2 \rangle$ and $\mathbf{w} = 4\widehat{\mathbf{i}} - 2\widehat{\mathbf{j}}$. (b) Are $\mathbf{v}, \mathbf{w}$ orthogonal?

(c) Find the angle θ_{vw} between $\mathbf{v}$ & $\mathbf{w}$.

EX 9.3.2: (a) Find the dot product $\mathbf{v} \cdot \mathbf{w}$ if $\mathbf{v} = \langle 1, 2, 3 \rangle$ and $\mathbf{w} = 4\widehat{\mathbf{i}} - 5\widehat{\mathbf{j}} - 6\widehat{\mathbf{k}}$. (b) Are $\mathbf{v}, \mathbf{w}$ orthogonal?

(c) Find the angle θ between $\mathbf{v}$ & $\mathbf{w}$. (d) Are $\mathbf{v}, \mathbf{w}$ parallel?

EX 9.3.3: Find scalar t such that vectors $\mathbf{v} = \langle 1, 3, t \rangle$ and $\mathbf{w} = 2t\widehat{\mathbf{i}} + \widehat{\mathbf{j}} - 4\widehat{\mathbf{k}}$ are orthogonal.

EX 9.3.4: Find the scalar & vector projections of $\mathbf{v} = \langle 1, 2 \rangle$ onto $\mathbf{w} = -3\widehat{\mathbf{i}} + 4\widehat{\mathbf{j}}$.

EX 9.3.5: Find the scalar & vector projections of $\mathbf{v} = \widehat{\mathbf{i}} + 2\widehat{\mathbf{j}} + 3\widehat{\mathbf{k}}$ onto $\mathbf{w} = \langle 3, -1, -1 \rangle$.

EX 9.3.6: Find the direction cosines and the direction angles (in radians) of the vector $\mathbf{v} = \langle 1, 2, 3 \rangle$.

EX 9.3.7: Find the work performed when the force $\mathbf{F} = \langle 1, 2, 3 \rangle$ moves an object along a straight line from point $P(-2, 3, 4)$ to point $Q(8, -7, -6)$.