VECTORS: CROSS PRODUCT [SST 9.4]

DETERMINANT OF A 2x2 MATRIX:

- Matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \implies \det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

CROSS PRODUCT OF TWO VECTORS:

- $\mathbf{v}, \mathbf{w} \in \mathbb{R}^3 \implies$ **Cross product** $\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \widehat{\mathbf{i}} & \widehat{\mathbf{j}} & \widehat{\mathbf{k}} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} \widehat{\mathbf{i}} - \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} \widehat{\mathbf{j}} + \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix} \widehat{\mathbf{k}}$
- Nonzero nonparallel vectors $\mathbf{v}, \mathbf{w} \in \mathbb{R}^3 \implies \mathbf{v} \times \mathbf{w}$ is orthogonal to both $\mathbf{v}$ & $\mathbf{w}$.
- $||\mathbf{v} \times \mathbf{w}|| = ||\mathbf{v}||\,||\mathbf{w}|| \sin(\theta_{vw})$, where the **angle between the two vectors** $\theta_{vw} \in [0, \pi]$ and nonzero vectors $\mathbf{v}, \mathbf{w} \in \mathbb{R}^3$
- RIGHT-HAND RULE:

  Point right arm along $\mathbf{v}$ & curl fingers toward $\mathbf{w}$ to cover angle θ_{vw}, then thumb points in the direction of $\mathbf{v} \times \mathbf{w}$.
- Notice that the cross product $\mathbf{v} \times \mathbf{w}$ is a vector **orthogonal to both vectors** $\mathbf{v}$ & $\mathbf{w}$.
- Nonzero vectors $\mathbf{v}, \mathbf{w}$ are **parallel** $\iff \mathbf{v} \times \mathbf{w} = \vec{\mathbf{0}}$

PROPERTIES OF CROSS PRODUCTS:

$\left(\text{Let vectors } \mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3 \text{ and scalars } s, t \in \mathbb{R}\right)$

- $(s\mathbf{v}) \times (t\mathbf{w}) = st(\mathbf{v} \times \mathbf{w})$
- $\mathbf{v} \times \vec{\mathbf{0}} = \vec{\mathbf{0}} \times \mathbf{v} = \vec{\mathbf{0}}$
- $\mathbf{v} \times \mathbf{w} = -(\mathbf{w} \times \mathbf{v})$
- $\mathbf{v} \times \mathbf{v} = \vec{\mathbf{0}}$
- $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})$
- $(\mathbf{u} + \mathbf{v}) \times \mathbf{w} = (\mathbf{u} \times \mathbf{w}) + (\mathbf{v} \times \mathbf{w})$
- $||\mathbf{v} \times \mathbf{w}||^2 = ||\mathbf{v}||^2 ||\mathbf{w}||^2 - (\mathbf{v} \cdot \mathbf{w})^2$
- $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{c} \cdot \mathbf{a})\mathbf{b} - (\mathbf{b} \cdot \mathbf{a})\mathbf{c}$

CROSS PRODUCTS & AREA:

- DEFINITION: Vertices $A, B, C \in \mathbb{R}^3$ are called **collinear** if **all three vertices lie on the same straight line**.
- Given a **parallelogram generated by nonzero nonparallel vectors** $\{\mathbf{AB}, \mathbf{AC}\}$, then:

  $$\boxed{\text{AREA OF PARALLELOGRAM}(\mathbf{AB}, \mathbf{AC}) = ||\mathbf{AB} \times \mathbf{AC}||}$$
- Given a **triangle generated by distinct noncollinear vertices** $\{A, B, C\}$, then:

  $$\boxed{\text{AREA OF TRIANGLE}(A, B, C) = \tfrac{1}{2}||\mathbf{AB} \times \mathbf{AC}||}$$

SCALAR TRIPLE PRODUCT & VOLUME:

- DEFINITION: Vertices $A, B, C, D \in \mathbb{R}^3$ are called **coplanar** if **all four vertices lie on the same plane**.
- DEFINITION: Vectors $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$ are called **coplanar** if **all three vectors lie in the same plane**.
- Given a **parallelopiped generated by nonzero noncoplanar vectors** $\{\mathbf{AB}, \mathbf{AC}, \mathbf{AD}\}$, then:

  $$\boxed{\text{VOLUME OF PARALLELOPIPED}(\mathbf{AB}, \mathbf{AC}, \mathbf{AD}) = |(\mathbf{AB} \times \mathbf{AC}) \cdot \mathbf{AD}|}$$
- Given a **tetrahedron generated by distinct noncoplanar vertices** $\{A, B, C, D\}$, then:

  $$\boxed{\text{VOLUME OF TETRAHEDRON}(A, B, C, D) = \tfrac{1}{6}|(\mathbf{AB} \times \mathbf{AC}) \cdot \mathbf{AD}|}$$

EX 9.4.1: Given vectors $\mathbf{v} = \langle 1, 2, 3 \rangle$ and $\mathbf{w} = 2\widehat{\mathbf{i}} - \widehat{\mathbf{j}} - \widehat{\mathbf{k}}$,

(a) Find $\mathbf{v} \times \mathbf{w}$.

(b) Find $\mathbf{w} \times \mathbf{v}$.

(c) Find the area of the parallelogram generated by vectors $\{\mathbf{v}, \mathbf{w}\}$.

EX 9.4.2: Given vectors $\mathbf{v} = \widehat{\mathbf{i}} - 2\widehat{\mathbf{k}}$ and $\mathbf{w} = \langle 0, 2, -2 \rangle$,

(a) Find $\mathbf{v} \times \mathbf{w}$.

(b) Find $[(2\mathbf{v}) \times (3\mathbf{w})] \cdot \left(4\widehat{\mathbf{k}}\right)$

(c) Find $\sin(\theta_{vw})$.

EX 9.4.3: Find the area of the triangle generated by vertices $A(0, -1, 2)$, $B(3, 2, 1)$, $C(5, 4, 6)$.

EX 9.4.4: Find the volume of the parallelopiped generated by vectors $\mathbf{u} = \langle 1, 2, 3 \rangle$, $\mathbf{v} = \langle 4, 0, -5 \rangle$, $\mathbf{w} = \langle 0, -3, 1 \rangle$.

EX 9.4.5: Let $\mathbf{v} = \langle 1, 2, 3 \rangle$ and $\mathbf{w} = \langle -3, 1, -2 \rangle$. Find a **unit vector** $\widehat{\mathbf{u}}$ such that $\widehat{\mathbf{u}}$ is orthogonal to both $\mathbf{v}$ & $\mathbf{w}$.

EX 9.4.6: Explain (briefly) why the following statement makes no sense: $(\mathbf{u} \times \mathbf{v}) - (\mathbf{u} \times \mathbf{u})$, where $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2$.

EX 9.4.7: Explain (briefly) why the following statement makes no sense: $(\mathbf{u} \times \mathbf{v}) - (\mathbf{u} \cdot \mathbf{v})$, where $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$.

EX 9.4.8: Explain (briefly) why the following statement makes no sense: $(\mathbf{u} \cdot \mathbf{v}) \times \mathbf{w}$, where $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$.