Area Between Curves

Calculus II

Josh Engwer

TTU

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Continuity & Differentiability of a Function (Notation)

Definition

Given function $f(x)$ and set $S \subseteq \mathbb{R}$. Then:

$$f \in C(S) \iff f \text{ is continuous on set } S$$

$$f \in C^1(S) \iff f, f' \in C(S) \implies f \text{ is differentiable on set } S$$

$$f \in C^2(S) \iff f, f', f'' \in C(S) \implies f \text{ is twice-differentiable on set } S$$

REMARK:

In general, f being differentiable on set S may not imply that $f \in C^1(S)$.

One such example is $f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & , \text{ if } x \neq 0 \\ 0 & , \text{ if } x = 0 \end{cases}$

Such "pathological functions" will not be considered in this course.

NOTATION:

$\mathbb{R} \equiv$ Interval $(-\infty, \infty) \equiv$ "The set of **real numbers**" $\equiv$ "The real line"

Definite Integrals (Definition & Interpretation)

Definition

(Riemann Sum Definition of an Integral)
Let $f \in C[a,b]$ where $[a,b]$ is a **closed interval** s.t. $-\infty < a < b < \infty$. Then:

$$\int_a^b f(x)\,dx := \lim_{N\to\infty} \sum_{k=1}^{N} f\left(x_k^*\right)\Delta x$$

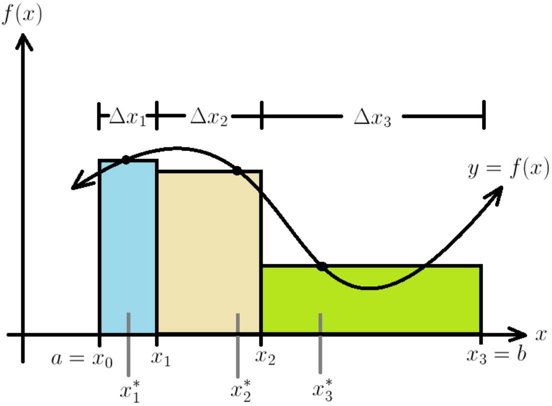
Proposition

(The Integral as an Area)
Let $f \in C[a,b]$ s.t. $f(x) \geq 0 \ \ \forall x \in [a,b]$. Then

$$\int_a^b f(x)\,dx$$

represents the ***area*** *of the region bounded by the curve $y = f(x)$, the x-axis, and the vertical lines $x = a$ & $x = b$.*

Riemann Sums (Non-Uniform, Arbitrary Tags)



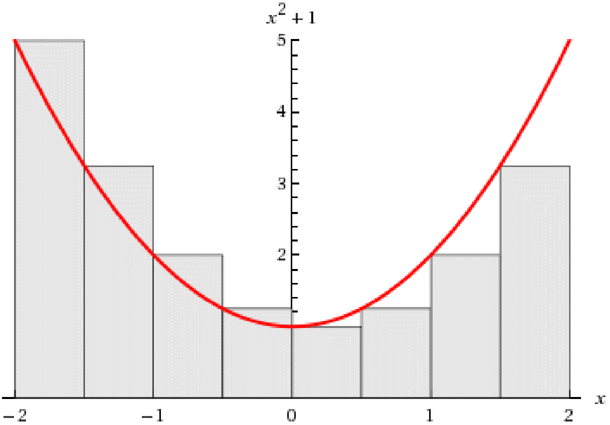
- # Rectangles : $N = 3$
- Partition : $\mathcal{P} = \{x_0, x_1, x_2, x_3\}$

Width of k^{th} Rectangle $= \Delta x_k$

Tags : $\mathcal{T} = \{x_1^*, x_2^*, x_3^*\}$

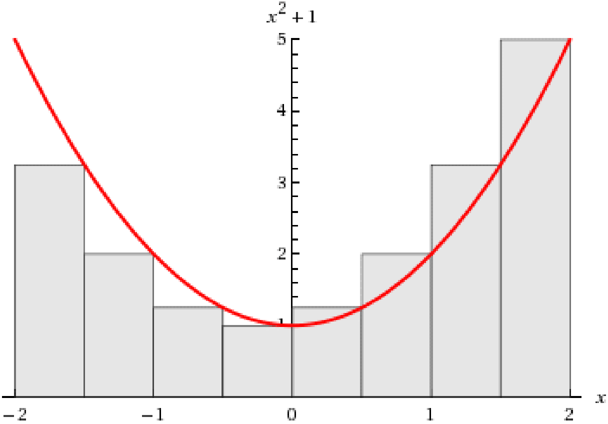
- Riemann Sum : $\displaystyle\sum_{k=1}^{N} f(x_k^*)\Delta x_k = f(x_1^*)\Delta x_1 + f(x_2^*)\Delta x_2 + f(x_3^*)\Delta x_3$

Riemann Sums (Uniform, Left-Endpoint Tags)



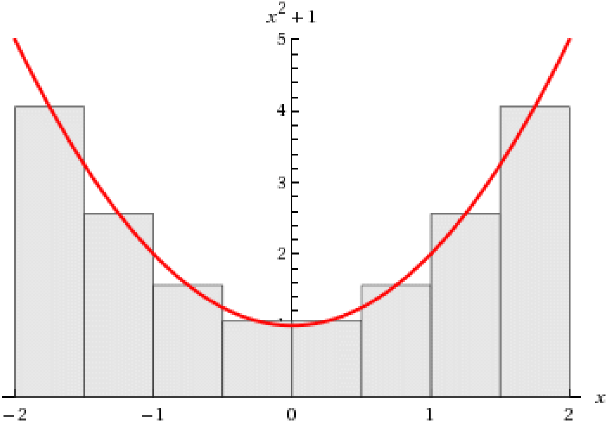
On the k^{th} subinterval $[x_{k-1}, x_k]$, let k^{th} tag $x_k^* := x_{k-1}$.

Riemann Sums (Uniform, Right-Endpoint Tags)



On the k^{th} subinterval $[x_{k-1}, x_k]$, let k^{th} tag $x_k^* := x_k$.

Riemann Sums (Uniform, Midpoint Tags)



On the k^{th} subinterval $[x_{k-1}, x_k]$, let k^{th} tag $x_k^* := \frac{1}{2}\left[x_{k-1} + x_k\right]$.

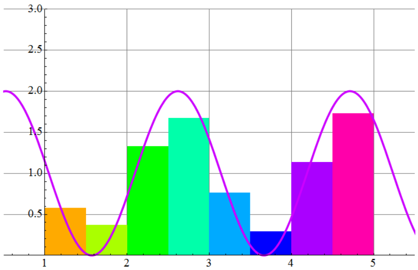
(DEMO) RIEMANN SUM DEFINITION OF AN INTEGRAL (Click below):

subdivision multiplier n 1

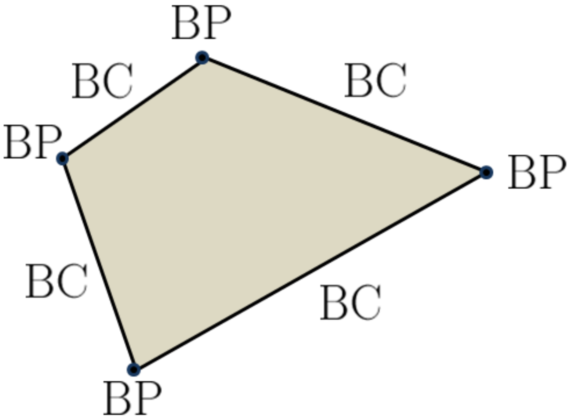
rectangle heights mean

RIEMANN SUMS by with enhancements by Josh Engwer (2013) [CC-BY-SA]

subdivisions of a unit interval	2	integral of $\sin(3x)+1$ from 1 to 5	3.92323
error	−0.0149645	Riemann sum	3.9382



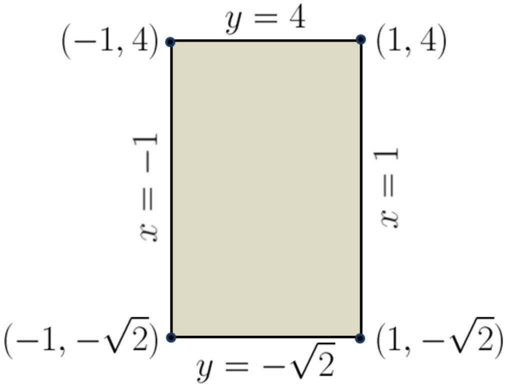
Boundary Curves (BC's) & Boundary Points (BP's)



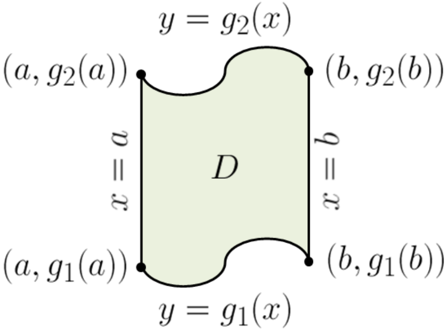
A region in $\mathbb{R}^2$ has **boundary points (BP's)** & **boundary curves (BC's)**.
A **boundary point (BP)** is the **intersection** of two **boundary curves (BC's)**.

<u>NOTATION:</u> $\mathbb{R}^2 \equiv xy$-plane.

Boundary Curves (BC's) & Boundary Points (BP's)



Vertically-Simple (V-Simple) Regions



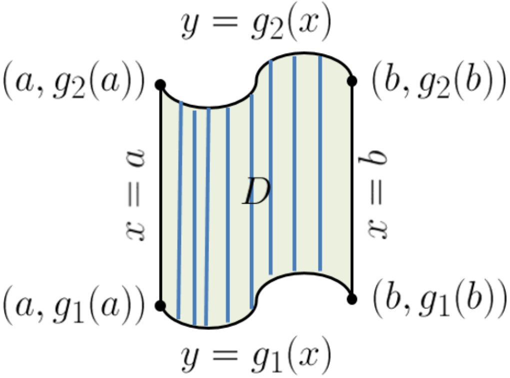
Vertically simple region

Definition

A region $D \subset \mathbb{R}^2$ is **vertically-simple (V-Simple)** if

the region has only one top BC & only one bottom BC.

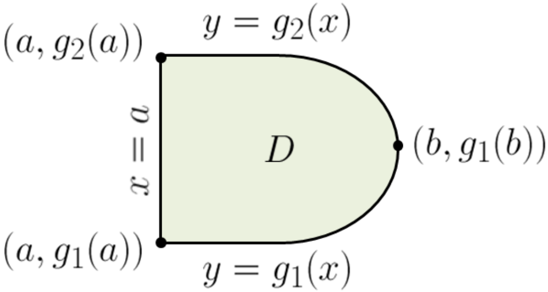
Vertically Simple (V-Simple) Regions (Definition)



Vertically simple region

i.e., V-Simple regions can be swept vertically (with vertical lines [in **blue**]) where each vertical line intersects the **same top BC** & **same bottom BC**.

Vertically-Simple (V-Simple) Regions



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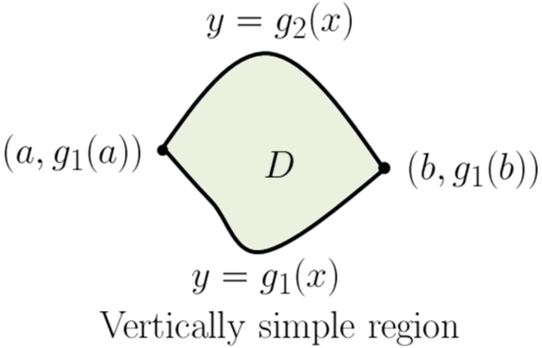

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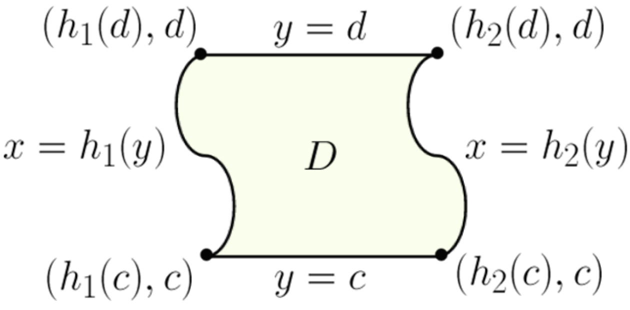
Horizontally-Simple (H-Simple) Regions



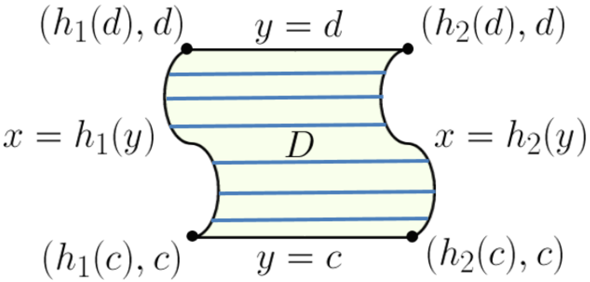
Horizontally simple region

Definition

A region $D \subset \mathbb{R}^2$ is **horizontally-simple (H-Simple)** if

the region has only one left BC & only one right BC.

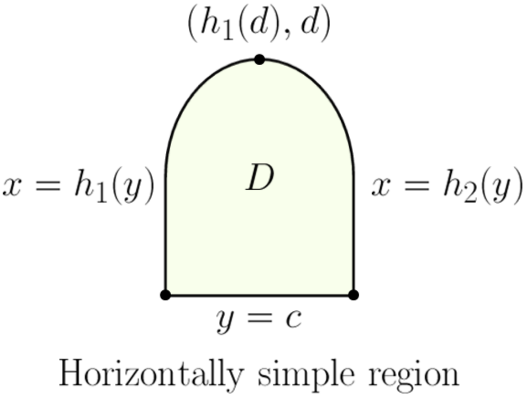
Horizontally Simple (H-Simple) Regions (Definition)



Horizontally simple region

i.e., H-Simple regions can be swept horizontally (w/ horizontal lines [in **blue**]) where each horizontal line intersects the **same left BC** & **same right BC**.

Horizontally-Simple (H-Simple) Regions



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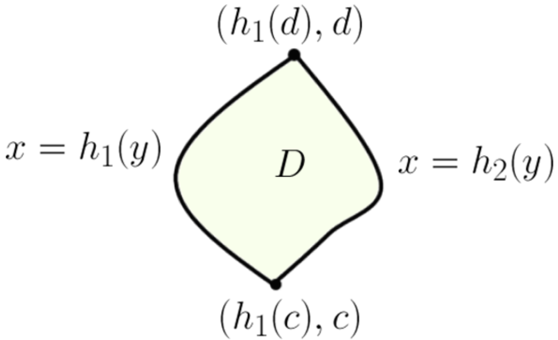
Horizontally-Simple (H-Simple) Regions



Horizontally simple region

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A region $D \subset \mathbb{R}^2$ is **horizontally-simple (H-Simple)** if

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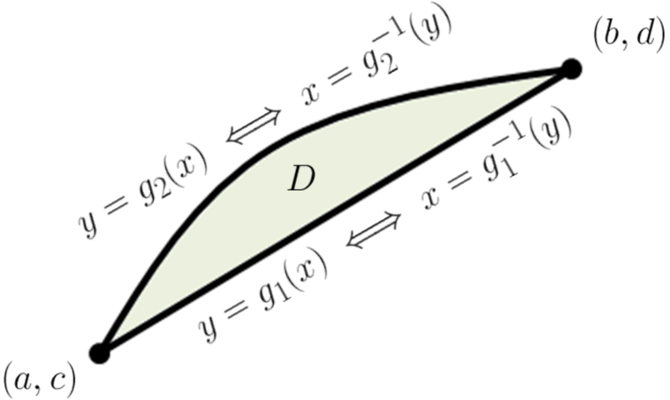
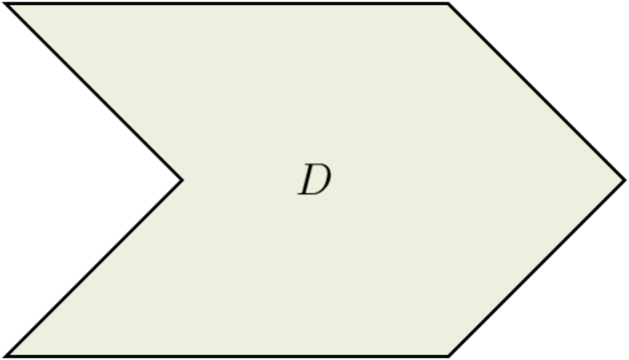
A Region that's both V-Simple & H-Simple



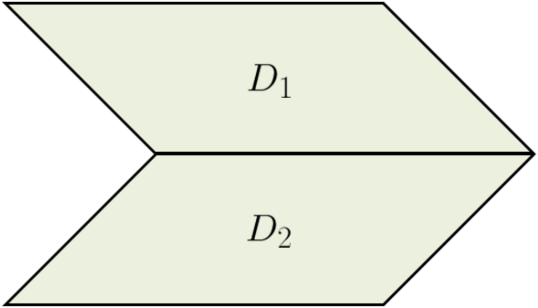
Both V-Simple & H-Simple Region

Area of a Region that's neither V-Simple nor H-Simple



Neither V-Simple Nor H-Simple Region

Area of a Region that's neither V-Simple nor H-Simple



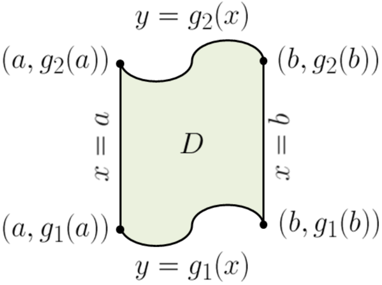
Subdivide Region into V-Simple and H-Simple Regions

$$D = D_1 \cup D_2$$

$$\text{Area}(D) = \text{Area}(D_1) + \text{Area}(D_2)$$

REMARK: Subdivide along a **BP** using a **horizontal** or **vertical line**.

Area of a V-Simple Region (Procedure)



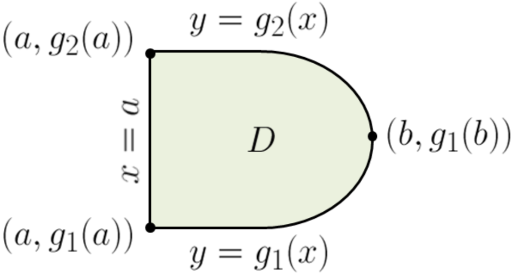
Vertically simple region

Proposition

(Area of a V-Simple Region)

$$\textit{Area}(D) = \int_{\text{smallest } x\text{-value in } D}^{\text{largest } x\text{-value in } D} \Big[(\textit{Top BC}) - (\textit{Bottom BC}) \Big] dx = \int_a^b \Big[g_2(x) - g_1(x) \Big] dx$$

Area of a V-Simple Region (Procedure)



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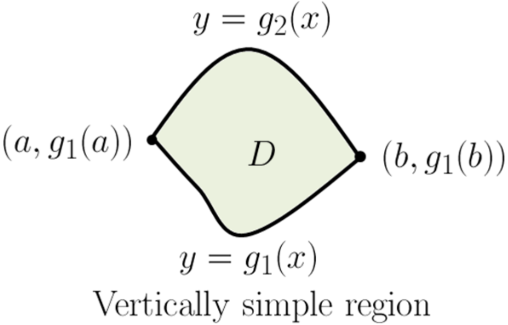

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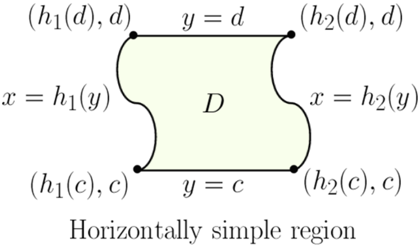
Area of a H-Simple Region (Procedure)

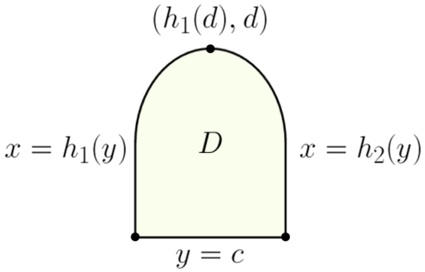


Proposition

(Area of a H-Simple Region)

$$\textit{Area}(D) = \int_{\textit{smallest } y\textit{-value in } D}^{\textit{largest } y\textit{-value in } D} \Big[(\textit{Right BC}) - (\textit{Left BC})\Big]\,dy = \int_c^d \Big[h_2(y) - h_1(y)\Big]\,dy$$

Area of a H-Simple Region (Procedure)



Horizontally simple region

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(Area of a H-Simple Region)

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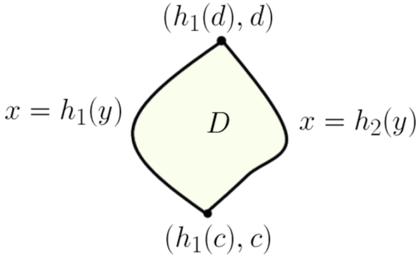

Horizontally simple region

Proposition

(Area of a H-Simple Region)

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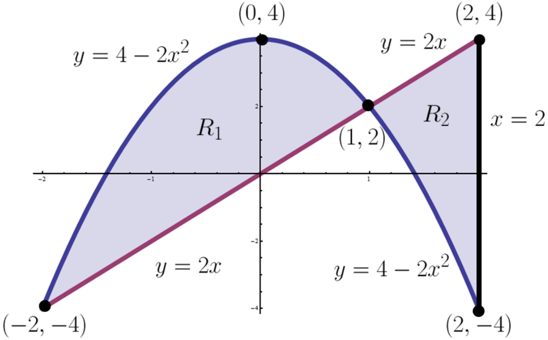
Area Between Two Curves (Using V-Rects)

EXAMPLE: Let region R be bounded by curves $y = 2x$, $y = 4 - 2x^2$, $x = 2$.
Setup integral(s) to compute Area(R) **using Vertical Rectangles (V-Rects)**.

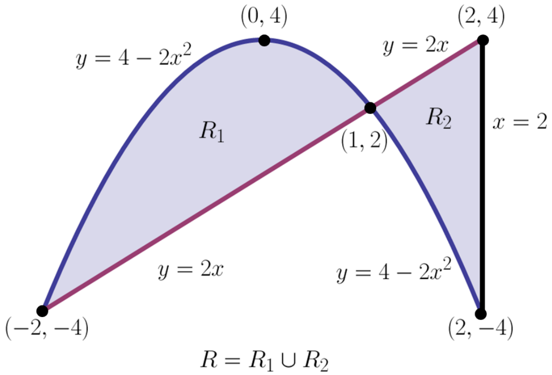
Area Between Two Curves (Using V-Rects)

EXAMPLE: Let region R be bounded by curves $y = 2x$, $y = 4 - 2x^2$, $x = 2$.



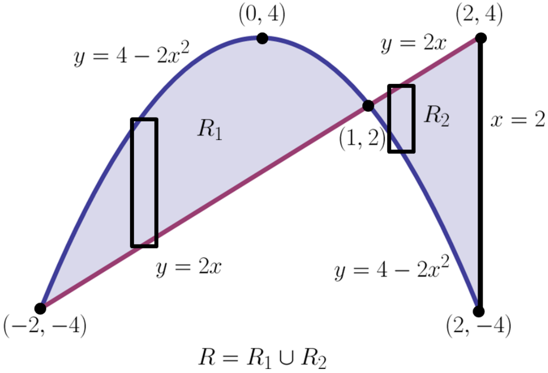
Sketch & characterize region R (label BP's & BC's in terms of x)
Notice subregions R_1, R_2 are each V-simple.

EXAMPLE: Let region R be bounded by curves $y = 2x$, $y = 4 - 2x^2$, $x = 2$.



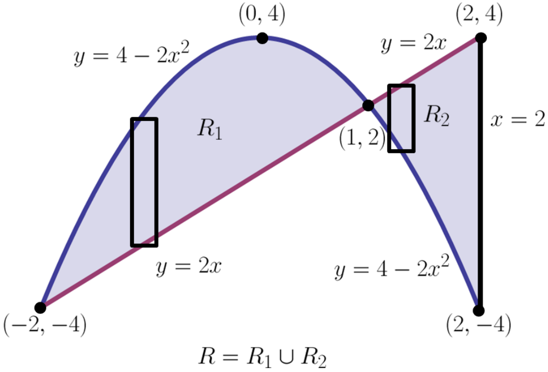
Sketch & characterize region R. (remove unnecessary clutter)
Notice subregions R_1, R_2 are each V-simple.

Area Between Two Curves (Using V-Rects)



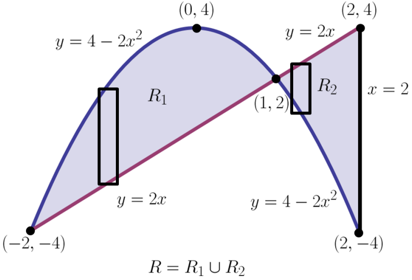
Key Element: V-Rectangle (V-Rect)

Area Between Two Curves (Using V-Rects)



k^{th} V-Rect in R_1:

$$\begin{array}{rcl} \text{Width} & = & \Delta x_k \\ \text{Height} & = & (\text{Top BC}) - (\text{Bottom BC}) \\ \hline \text{Area} & = & (\text{Height}) \times (\text{Width}) \end{array}$$

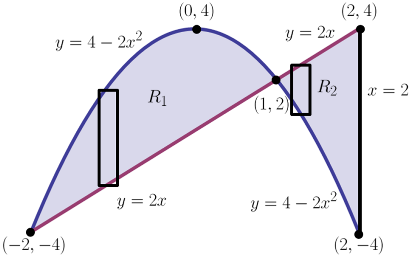


k^{th} V-Rect in R_1:

$$\begin{array}{lcl} \text{Width} & = & \Delta x_k \\ \text{Height} & = & \left[4 - 2\left(x_k^*\right)^2\right] - 2x_k^* \\ \hline \text{Area} & = & \left[4 - 2x_k^* - 2\left(x_k^*\right)^2\right] \Delta x_k \end{array}$$

Riemann Sum: Area(R_1) $\approx A_N^* = \sum_{k=1}^{N} \left[4 - 2x_k^* - 2\left(x_k^*\right)^2\right] \Delta x_k$

Area Between Two Curves (Using V-Rects)

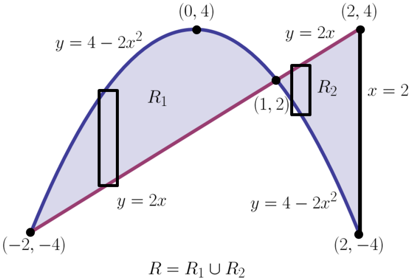


$R = R_1 \cup R_2$

Riemann Sum: Area$(R_1) \approx A_N^* = \sum_{k=1}^{N} \left[4 - 2x_k^* - 2\left(x_k^*\right)^2\right] \Delta x_k$

Integral: Area$(R_1) = \lim_{N\to\infty} A_N^* = \int_{\text{smallest } x\text{-coord. in } R_1}^{\text{largest } x\text{-coord. in } R_1} \left(4 - 2x - 2x^2\right)\, dx$

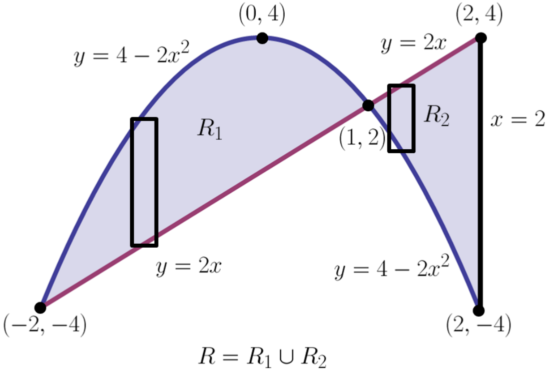
Area Between Two Curves (Using V-Rects)



Riemann Sum: Area$(R_1) \approx A_N^* = \sum_{k=1}^{N} \left[4 - 2x_k^* - 2\left(x_k^*\right)^2\right] \Delta x_k$

Integral: Area$(R_1) = \lim_{N\to\infty} A_N^* = \int_{-2}^{1} \left(4 - 2x - 2x^2\right)\, dx$

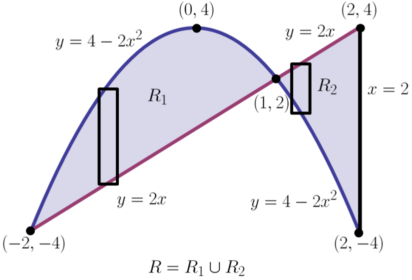
Area Between Two Curves (Using V-Rects)



k^{th} V-Rect in R_2:

Width	=	Δx_k
Height	=	(Top BC) − (Bottom BC)
Area	=	(Height) × (Width)

Area Between Two Curves (Using V-Rects)

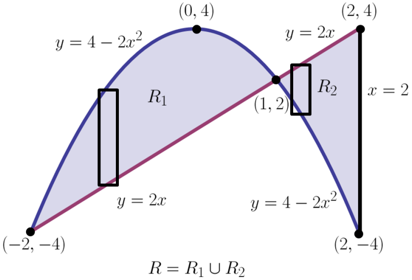


k^{th} V-Rect in R_2:

$$\begin{array}{lcl} \text{Width} & = & \Delta x_k \\ \text{Height} & = & 2x_k^* - \left[4 - 2\left(x_k^*\right)^2\right] \\ \hline \text{Area} & = & \left[2\left(x_k^*\right)^2 + 2x_k^* - 4\right]\Delta x_k \end{array}$$

Riemann Sum: Area$(R_2) \approx A_N^* = \sum_{k=1}^{N}\left[2\left(x_k^*\right)^2 + 2x_k^* - 4\right]\Delta x_k$

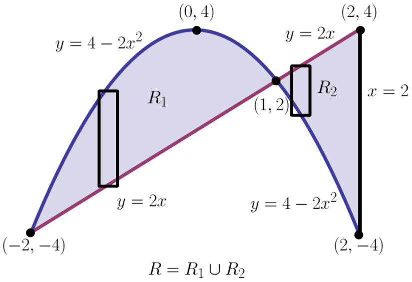
Area Between Two Curves (Using V-Rects)



Riemann Sum: Area(R_2) $\approx A_N^* = \displaystyle\sum_{k=1}^{N} \left[2\left(x_k^*\right)^2 + 2x_k^* - 4\right] \Delta x_k$

Integral: Area(R_2) $= \displaystyle\lim_{N\to\infty} A_N^* = \int_{\text{smallest } x\text{-coord. in } R_2}^{\text{largest } x\text{-coord. in } R_2} \left(2x^2 + 2x - 4\right)\, dx$

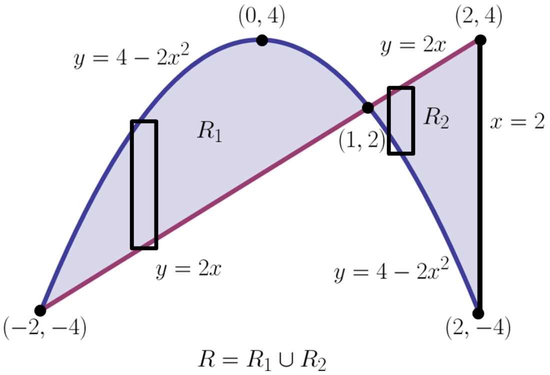
Area Between Two Curves (Using V-Rects)



Riemann Sum: Area(R_2) $\approx A_N^* = \sum_{k=1}^{N} \left[2\left(x_k^*\right)^2 + 2x_k^* - 4\right] \Delta x_k$

Integral: Area(R_2) $= \lim_{N\to\infty} A_N^* = \int_1^2 \left(2x^2 + 2x - 4\right)\, dx$

Area Between Two Curves (Using V-Rects)

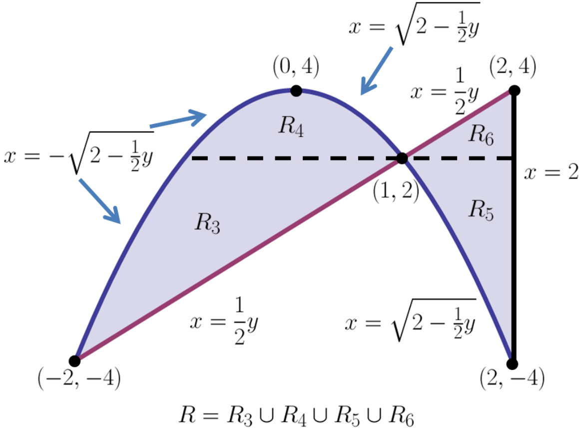


Area(R) = Area(R_1) + Area(R_2)

$$= \boxed{\int_{-2}^{1} \left(4 - 2x - 2x^2\right)\ dx + \int_{1}^{2} \left(2x^2 + 2x - 4\right)\ dx} = 9 + \frac{11}{3} = \frac{38}{3}$$

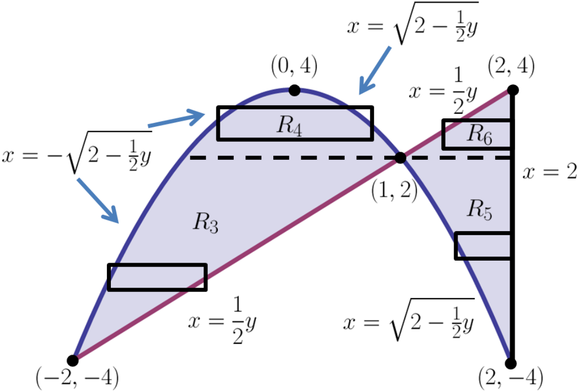
Area Between Two Curves (Using H-Rects)

EXAMPLE: Let region R be bounded by curves $y = 2x$, $y = 4 - 2x^2$, $x = 2$.
Setup integral(s) to compute Area(R) **using H-Rects**.



Sketch & characterize region R (label BP's & BC's in terms of y)
Subdivide region (via dashed line) into four H-simple subregions R_3, R_4, R_5, R_6.

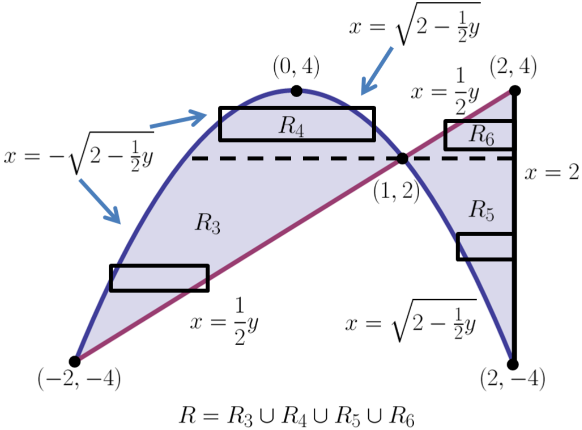
Area Between Two Curves (Using H-Rects)



$$R = R_3 \cup R_4 \cup R_5 \cup R_6$$

Key Element: H-Rectangle (H-Rect)

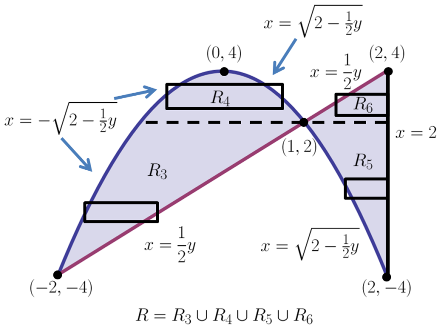
Area Between Two Curves (Using H-Rects)



k^{th} H-Rect in R_3:

Width	=	Δy_k
Length	=	(Right BC) − (Left BC)
Area	=	(Length) × (Width)

Area Between Two Curves (Using H-Rects)

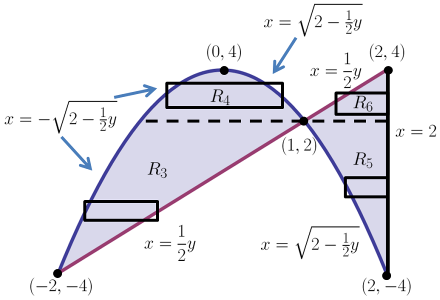


k^{th} H-Rect in R_3:

$$\begin{aligned} \text{Width} &= \Delta y_k \\ \text{Length} &= \tfrac{1}{2}y_k^* - \left(-\sqrt{2 - \tfrac{1}{2}y_k^*}\right) \\ \hline \text{Area} &= \left(\tfrac{1}{2}y_k^* + \sqrt{2 - \tfrac{1}{2}y_k^*}\right)\Delta y_k \end{aligned}$$

Riemann Sum: $\text{Area}(R_3) \approx A_N^* = \sum_{k=1}^{N} \left(\frac{1}{2}y_k^* + \sqrt{2 - \frac{1}{2}y_k^*}\right)\Delta y_k$

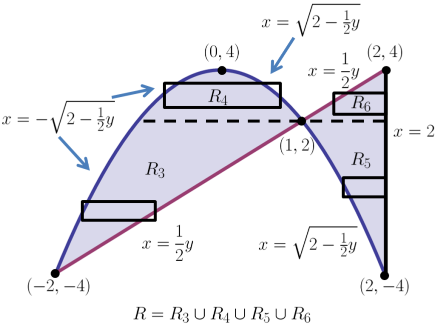
Area Between Two Curves (Using H-Rects)



Riemann Sum: $\text{Area}(R_3) \approx A_N^* = \displaystyle\sum_{k=1}^{N} \left(\frac{1}{2}y_k^* + \sqrt{2 - \frac{1}{2}y_k^*} \right) \Delta y_k$

Integral: $\text{Area}(R_3) = \displaystyle\lim_{N \to \infty} A_N^* = \int_{\text{smallest } y\text{-coord. in } R_3}^{\text{largest } y\text{-coord. in } R_3} \left(\frac{1}{2}y + \sqrt{2 - \frac{1}{2}y} \right) dy$

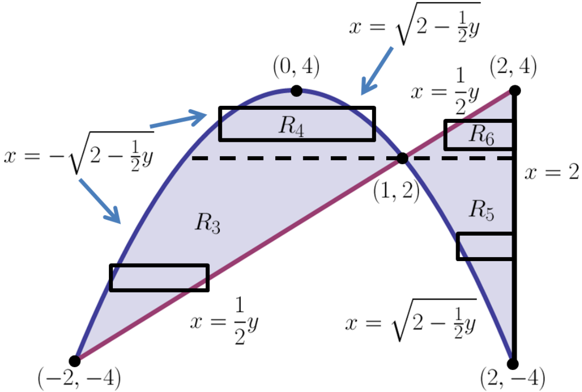
Area Between Two Curves (Using H-Rects)



Riemann Sum: $\text{Area}(R_3) \approx A_N^* = \sum_{k=1}^{N} \left(\frac{1}{2}y_k^* + \sqrt{2 - \frac{1}{2}y_k^*} \right) \Delta y_k$

Integral: $\text{Area}(R_3) = \lim_{N\to\infty} A_N^* = \int_{-4}^{2} \left(\frac{1}{2}y + \sqrt{2 - \frac{1}{2}y} \right) \, dy$

Area Between Two Curves (Using H-Rects)

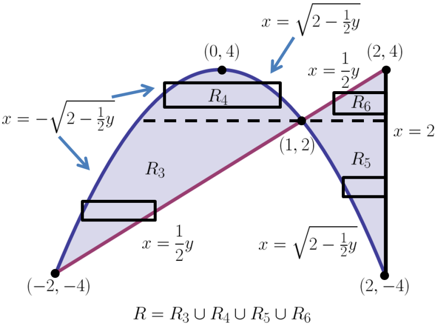


$$R = R_3 \cup R_4 \cup R_5 \cup R_6$$

k^{th} H-Rect in R_4:

Width	=	Δy_k
Length	=	(Right BC) − (Left BC)
Area	=	(Length) × (Width)

Area Between Two Curves (Using H-Rects)



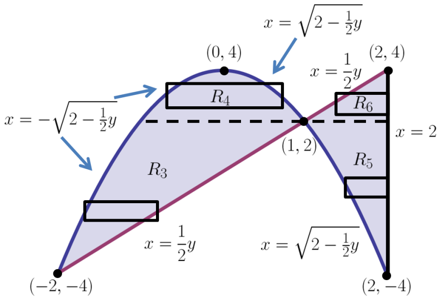
$R = R_3 \cup R_4 \cup R_5 \cup R_6$

k^{th} H-Rect in R_4:

$$\begin{aligned} \text{Width} &= \Delta y_k \\ \text{Length} &= \sqrt{2 - \tfrac{1}{2}y_k^*} - \left(-\sqrt{2 - \tfrac{1}{2}y_k^*}\right) \\ \hline \text{Area} &= 2\sqrt{2 - \tfrac{1}{2}y_k^*}\,\Delta y_k \end{aligned}$$

Riemann Sum: Area(R_4) $\approx A_N^* = \sum_{k=1}^{N} 2\sqrt{2 - \frac{1}{2}y_k^*}\,\Delta y_k$

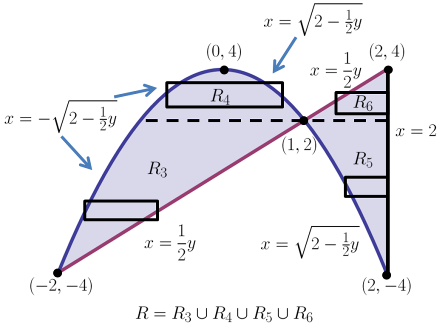
Area Between Two Curves (Using H-Rects)



Riemann Sum: Area(R_4) $\approx A_N^* = \displaystyle\sum_{k=1}^{N} 2\sqrt{2 - \frac{1}{2}y_k^*}\,\Delta y_k$

Integral: Area(R_4) $= \displaystyle\lim_{N\to\infty} A_N^* = \int_{\text{smallest } y\text{-coord. in } R_4}^{\text{largest } y\text{-coord. in } R_4} 2\sqrt{2 - \frac{1}{2}y}\,dy$

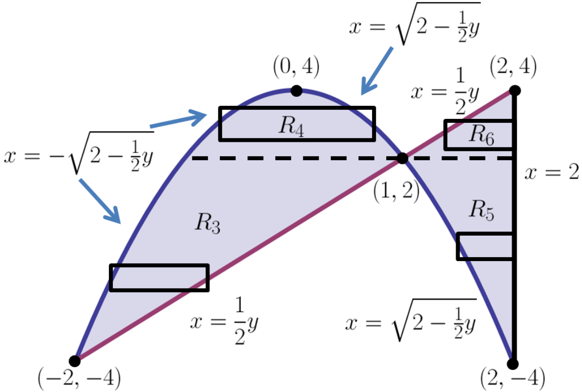
Area Between Two Curves (Using H-Rects)



Riemann Sum: Area(R_4) $\approx A_N^* = \sum_{k=1}^{N} 2\sqrt{2 - \frac{1}{2}y_k^*}\,\Delta y_k$

Integral: Area(R_4) $= \lim_{N\to\infty} A_N^* = \int_2^4 2\sqrt{2 - \frac{1}{2}y}\,dy$

Area Between Two Curves (Using H-Rects)

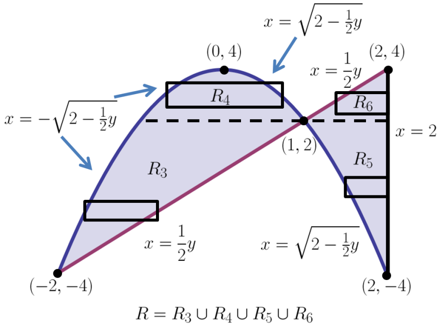


$$R = R_3 \cup R_4 \cup R_5 \cup R_6$$

k^{th} H-Rect in R_5:

Width	=	Δy_k
Length	=	(Right BC) − (Left BC)
Area	=	(Length) × (Width)

Area Between Two Curves (Using H-Rects)

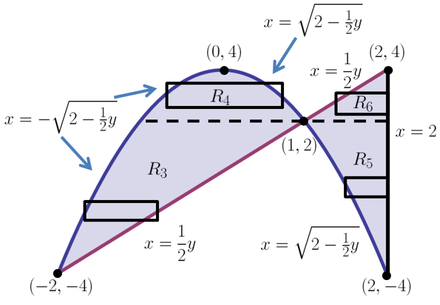


k^{th} H-Rect in R_5:

$$\begin{aligned} \text{Width} &= \Delta y_k \\ \text{Length} &= 2 - \sqrt{2 - \tfrac{1}{2}y_k^*} \\ \hline \text{Area} &= \left(2 - \sqrt{2 - \tfrac{1}{2}y_k^*}\right) \Delta y_k \end{aligned}$$

Riemann Sum: Area$(R_5) \approx A_N^* = \sum_{k=1}^{N} \left(2 - \sqrt{2 - \frac{1}{2}y_k^*}\right) \Delta y_k$

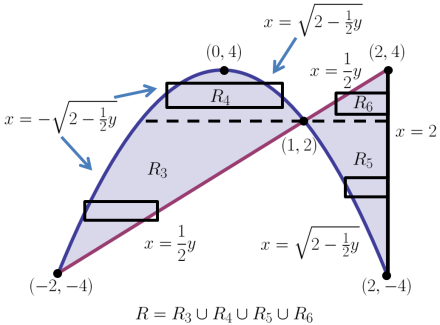
Area Between Two Curves (Using H-Rects)



Riemann Sum: $\text{Area}(R_5) \approx A_N^* = \sum_{k=1}^{N} \left(2 - \sqrt{2 - \frac{1}{2}y_k^*}\right) \Delta y_k$

Integral: $\text{Area}(R_5) = \lim_{N\to\infty} A_N^* = \int_{\text{smallest } y\text{-coord. in } R_5}^{\text{largest } y\text{-coord. in } R_5} \left(2 - \sqrt{2 - \frac{1}{2}y}\right)\, dy$

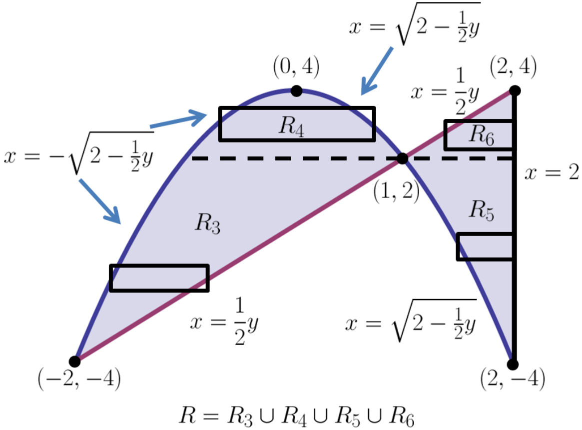
Area Between Two Curves (Using H-Rects)



Riemann Sum: Area$(R_5) \approx A_N^* = \sum_{k=1}^{N} \left(2 - \sqrt{2 - \frac{1}{2}y_k^*}\right) \Delta y_k$

Integral: Area$(R_5) = \lim_{N\to\infty} A_N^* = \int_{-4}^{2} \left(2 - \sqrt{2 - \frac{1}{2}y}\right)\, dy$

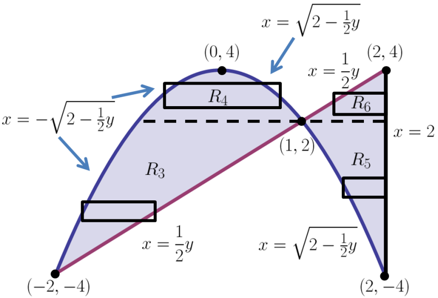
Area Between Two Curves (Using H-Rects)



k^{th} H-Rect in R_6:

Width	$=$	Δy_k
Length	$=$	(Right BC) $-$ (Left BC)
Area	$=$	(Length) $\times$ (Width)

Area Between Two Curves (Using H-Rects)



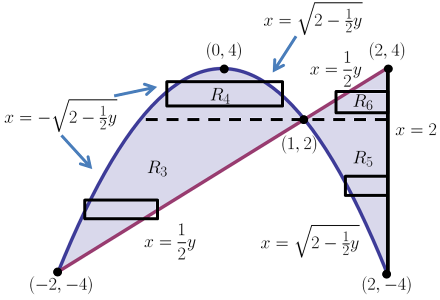
$R = R_3 \cup R_4 \cup R_5 \cup R_6$

k^{th} H-Rect in R_6:

$$\begin{array}{lcl} \text{Width} & = & \Delta y_k \\ \text{Length} & = & 2 - \frac{1}{2}y_k^* \\ \hline \text{Area} & = & \left(2 - \frac{1}{2}y_k^*\right)\Delta y_k \end{array}$$

Riemann Sum: Area$(R_6) \approx A_N^* = \sum_{k=1}^{N} \left(2 - \frac{1}{2}y_k^*\right)\Delta y_k$

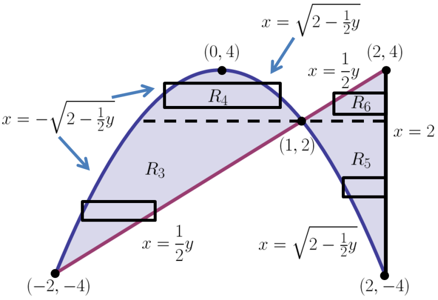
Area Between Two Curves (Using H-Rects)



Riemann Sum: Area(R_6) $\approx A_N^* = \sum_{k=1}^{N} \left(2 - \frac{1}{2}y_k^*\right) \Delta y_k$

Integral: Area(R_6) $= \lim_{N \to \infty} A_N^* = \int_{\text{smallest } y\text{-coord. in } R_6}^{\text{largest } y\text{-coord. in } R_6} \left(2 - \frac{1}{2}y\right)\, dy$

Area Between Two Curves (Using H-Rects)

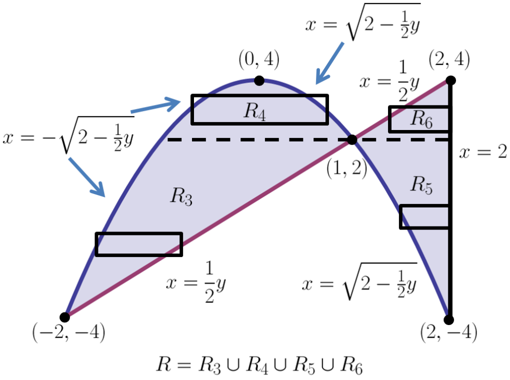


$R = R_3 \cup R_4 \cup R_5 \cup R_6$

Riemann Sum: Area$(R_6) \approx A_N^* = \displaystyle\sum_{k=1}^{N} \left(2 - \sqrt{2 - \frac{1}{2}y_k^*}\right) \Delta y_k$

Integral: Area$(R_6) = \displaystyle\lim_{N\to\infty} A_N^* = \int_2^4 \left(2 - \frac{1}{2}y\right)\, dy$

Area Between Two Curves (Using H-Rects)



Area(R) = Area(R_3) + Area(R_4) + Area(R_5) + Area(R_6)

$$= \int_{-4}^{2} \left(\frac{1}{2}y + \sqrt{2 - \frac{1}{2}y} \right) \, dy + \int_{2}^{4} 2\sqrt{2 - \frac{1}{2}y} \, dy + \int_{-4}^{2} \left(2 - \sqrt{2 - \frac{1}{2}y} \right) \, dy + \int_{2}^{4} \left(2 - \frac{1}{2}y \right) \, dy = \frac{19}{3} + \frac{8}{3} + \frac{8}{3} + 1 = \frac{38}{3}$$

The Facts of Life (according to WeBWorK)

".....and then WeBWorK decreed:"

- The Good News: Many HW problems just want the integral(s) setup.
- The Bad News: Many HW problems want the integral(s) computed.

So let's briefly recap computation of basic integrals from Calculus I....

Indefinite Integral Rules (from Calculus I)

Here, $C \in \mathbb{R}$ is called the **constant of integration**.
Also, $k \in \mathbb{R}$.

Zero Rule: $$\int 0 \; dx = C$$

Constant Rule: $$\int k \; dx = kx + C$$

Constant Multiple Rule: $$\int kf(x) \; dx = k \int f(x) \; dx$$

Sum/Diff Rule: $$\int [f(x) \pm g(x)] \; dx = \int f(x) \; dx \pm \int g(x) \; dx$$

Power Rule: $$\int x^n \; dx = \frac{1}{n+1} x^{n+1} + C \qquad \Big(\text{provided } n \in \mathbb{R} \setminus \{-1\}\Big)$$

$$\int e^x \; dx = e^x + C$$

$$\int a^x \; dx = \frac{a^x}{\ln a} + C \qquad \Big(\text{provided } a \in \mathbb{R}_+ \setminus \{1\}\Big)$$

$$\int \frac{1}{x} \; dx = \ln |x| + C$$

<u>NOTATION:</u> $\mathbb{R} \setminus \{-1\}$ means "All real numbers **except** -1"
<u>NOTATION:</u> $\mathbb{R}_+ \setminus \{1\}$ means "All **positive** real numbers **except** 1"

Indefinite Integral Rules (from Calculus I)

Here, $C \in \mathbb{R}$ is called the **constant of integration**.

- $\int \sin x \, dx = -\cos x + C$
- $\int \cos x \, dx = \sin x + C$
- $\int \sec^2 x \, dx = \tan x + C$
- $\int \sec x \tan x \, dx = \sec x + C$
- $\int \csc^2 x \, dx = -\cot x + C$
- $\int \csc x \cot x \, dx = -\csc x + C$
- $\int \frac{1}{\sqrt{1 - x^2}} \, dx = \arcsin x + C$
- $\int \frac{1}{1 + x^2} \, dx = \arctan x + C$
- $\int \frac{1}{|x|\sqrt{x^2 - 1}} \, dx = \text{arcsec}\, x + C$

Fundamental Theorem of Calculus (FTC)

Theorem

Let function $f \in C^1[a,b]$. *Then* $$\int_a^b f'(x)\,dx = f(b) - f(a)$$

WORKED EXAMPLE: Compute $I = \displaystyle\int_1^2 \left(x^4 - 3\right)\,dx$.

$$\int_1^2 \left(x^4 - 3\right)\,dx = \left[\frac{1}{5}x^5 - 3x\right]_{x=1}^{x=2} \stackrel{FTC}{=} \left[\frac{1}{5}(2)^5 - 3(2)\right] - \left[\frac{1}{5}(1)^5 - 3(1)\right] = \boxed{\frac{16}{5}}$$

WORKED EXAMPLE: Compute $I = \displaystyle\int_{\pi/6}^{3\pi/4} \sin\theta\,d\theta$.

$$\begin{aligned}\int_{\pi/6}^{3\pi/4} \sin\theta\,d\theta &= \Big[-\cos\theta\Big]_{\theta=\pi/6}^{\theta=3\pi/4} \stackrel{FTC}{=} [-\cos(3\pi/4)] - [-\cos(\pi/6)] \\ &= -\left(-\frac{\sqrt{2}}{2}\right) + \left(\frac{\sqrt{3}}{2}\right) = \boxed{\frac{\sqrt{2}+\sqrt{3}}{2}}\end{aligned}$$

Change of Variables (u-Substitution)

WORKED EXAMPLE: Evaluate $I = \displaystyle\int xe^{x^2}\,dx$.

CV: Let $u = x^2$, then $du = 2x\,dx \implies x\,dx = \frac{1}{2}\,du$

$$\implies \int xe^{x^2}\,dx \overset{CV}{=} \int e^u \left(\frac{1}{2}\,du\right) = \frac{1}{2}e^u + C \overset{CV}{=} \boxed{\frac{1}{2}e^{x^2} + C}$$

WORKED EXAMPLE: Evaluate $I = \displaystyle\int_{-2}^{3} xe^{x^2}\,dx$.

CV: Let $u = x^2$, then $du = 2x\,dx \implies x\,dx = \frac{1}{2}\,du$
and $u(-2) = (-2)^2 = 4$ and $u(3) = (3)^2 = 9$

$$\implies \int_{-2}^{3} xe^{x^2}\,dx \overset{CV}{=} \int_{4}^{9} e^u \left(\frac{1}{2}\,du\right) = \left[\frac{1}{2}e^u\right]_{u=4}^{u=9} \overset{FTC}{=} \boxed{\frac{1}{2}\left(e^9 - e^4\right)}$$

Nonelementary Integrals

Definition

A **nonelementary integral** is an integral whose antiderivative cannot be expressed in a finite closed form.

Here's a small list of nonelementary integrals (there are many, many more):

$$\int e^{x^2}\,dx \qquad \int \frac{e^x}{x}\,dx \qquad \int \sqrt{x}e^{-x}\,dx$$

$$\int \sin(x^2)\,dx \qquad \int \cos(e^x)\,dx \qquad \int e^{\cos x}\,dx$$

$$\int \sqrt{1+x^4}\,dx \qquad \int \ln(\ln x)\,dx \qquad \int \frac{x}{e^x-1}\,dx$$

$$\int \frac{1}{\ln x}\,dx \qquad \int \frac{\sin x}{x}\,dx \qquad \int \sin(\sin x)\,dx$$

$$\int x^x\,dx \qquad \int \frac{1}{x^x}\,dx \qquad \int \arctan(\ln x)\,dx$$

- When computing integrals, avoid nonelementary integrals!
- If using V-Rects leads to a nonelementary integral, use H-Rects instead.
- If using H-Rects leads to a nonelementary integral, use V-Rects instead.

Fin.